Long-time existence and growth of Sobolev norms for solutions of semi-linear Klein-Gordon equations and linear Schrödinger equations on some manifolds

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Abstract

This thesis is made of two parts, dealing respectively with semi-linear Klein-Gordon equations and linear Schrödinger equations.

In recent years, a number of results have been obtained on the question of long-time existence for solutions of semi-linear evolution equations (like Schrödinger or Klein-Gordon) on compact manifolds, with small smooth Cauchy data. Unlike in the case of the euclidean space, one cannot exploit dispersion to prove lower bounds for the time of existence of solutions in function of the size $\epsilon \to 0$ of the data. Instead, one uses normal form methods. These methods are effective, in most previous works, only when the spectrum of the Laplace-Beltrami operator on the compact manifolds at hand satisfies a very special condition of "increasing gap" between successive eigenvalues.

The first chapter of this thesis examines similar problems for models for which such an " increasing gap assumption " is not valid. We first consider a non-linear Klein-Gordon equation whose linear part is given in terms of the harmonic oscillator $-\Delta+|x|^2$ on $\mathbb{R}^d$. The spectrum of this operator is discrete (which reflects that, as in the case of compact manifolds, the solution of the corresponding linear Klein-Gordon equation does not disperse when time goes to infinity). The gap between successive eigenvalues is no longer increasing, but constant. Nevertheless, we are able to implement a normal form method that allows us to show that, for almost all masses, solutions of the evolution equation with small data exist for a larger time than the one which is given by local existence theory.

We apply next the same kind of ideas to the non-linear Klein-Gordon equation on the torus, for which we have also non increasing gaps between eigenvalues. We obtain a lower bound for the time of existence of small solutions, for almost every value of the mass, that provides an improvement of the results of Delort [14] in higher dimensions.

The second chapter of the thesis deals with linear Schrödinger equations on the torus with time dependent potential $V(x,t)$. It had been proved by Bourgain [8] that when V is smooth and bounded, as well as all its derivatives, the Sobolev norm of the solutions of that operator enjoy $O(|t|^\epsilon)$-bounds when $|t| \to +\infty$, for any $\epsilon > 0$. Wei-Min Wang showed, in one dimension, and for analytic potential, that one may prove logarithmic bounds $O(|\log t|^{\zeta s})$, $t \to +\infty$ for the H^s-norm. In the last paper of thesis, we adapt the method of Delort [15] to obtain such logarithmic estimates on tori of any dimension, under Gevrey regularity assumptions on the potential. We are able to express the exponent ζ in terms of the Gevrey index.

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Chapter 0 Introduction

Let $-\Delta + |x|^2$ be the harmonic oscillator on $\mathbb{R}^d$. The first section of Chapter 1 is devoted to the proof of lower bounds for the existence time of solutions of non-linear Klein-Gordon equations of type

$$(\partial_t^2 - \Delta + |x|^2 + m^2)v = v^{\kappa+1} \tag{1}$$

on $\mathbb{R}^d \times \mathbb{R}$ with small Cauchy data in some Sobolev space.

Let us recall some known results for a similar equation

$$(\partial_t^2 - \Delta + m^2)v = F(v, \partial_t v, \partial_x v) \tag{2}$$

on $\mathbb{R}^d \times \mathbb{R}$ (i.e., there is no quadratic potential in equation (1) and the nonlinearity is more general) when Cauchy data are smooth, rapidly decaying and of size $\epsilon \ll 1$. It has been proved independently by Klainermann [31] and Shatal [41] that the equation (in fact, an equation which contains more general nonlinearities) has a global solution when the dimension $d \geq 3$. Their proofs rely on the use of dispersive properties of the linear Klein-Gordon equation. In these dimensions, the nonlinearity can be considered as a short range perturbation, if it is at least of order 2 at the origin, since the solution of the linear equation decays at an integrable rate $t^{-\frac{d}{2}}$ when t goes to infinity. On the other hand, when $d = 2$ and the nonlinearity is quadratic, the decay rate $t^{-\frac{d}{2}}$ is no longer integrable. So the perturbation is long range and the situation is more delicate. However, global existence was proved by Georgiev et al. [26] and Kosecki [33] for special nonlinearities and later by Ozawa et al. [40] for general nonlinearities (see also Simon et al. [42] and Delort et al. [17]). A key point was to use, in combination with dispersive properties of the equation, the normal form method introduced initially by Shatah [41] in higher dimensions. The idea is to find a quadratic perturbation such that the action of the Klein-Gordon operator $\partial_t^2 - \Delta + m^2$ on it cancels out the quadratic part of the nonlinearity up to a remainder of higher order. Thus the problem is reduced to the cubic nonlinearity case, which was already treated. In one space dimension, it was first conjectured by Hörmander [28], [29] that the time of existence of the solution had an exponential lower bound e^{c/ϵ^2} for some positive constant c. Later, a proof was given by Moriyama et al. [38]. The optimality of this result follows by an example constructed in Keel et al. [30], but global existence was proved in Delort [13] when the nonlinearity satisfies a special condition (a "null condition" in the terminology introduced by Klainerman in the case of the wave equation in three space dimensions [32]). Let us also mention that in the case of nonlinearities which are not smooth functions of their arguments, lower bounds for the time of existence have been obtained by Lindbland et al. [37].

In most of the preceding works, a fairly rapid decay of Cauchy data at infinity is assumed. This assumption plays an essential role in the proofs. A natural question is to examine what we may get when the data just decay weakly as H^s

function. This question was first addressed in Delort [12], where periodic Cauchy data were studied. One gets therein a lower bound of the time of existence, of magnitude $c\epsilon^{-2}$ as well as examples showing the optimality of that result. For the equation (2) on $\mathbb{R}^d \times \mathbb{R}$, when $d = 1$ and the initial data are only assumed to be in $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ and of size $\epsilon \ll 1$ with s a large integer, it has been showed in Delort [11], that the solution exists on an interval of length $c\epsilon^{-4}|\log \epsilon|^{-6}$. Later, the same kind results were extended to $d \geq 2$ in Delort et al. [16]. The authors showed that the solution exists on an interval of time of length $\exp[c\epsilon^{-\mu}]$ with $\mu = 2/3$ if $d = 2$, $\mu = 1$ if $d \geq 3$.

Once some results for a single equation are obtained, one may expect to see what happens when a system is considered. For positive numbers m_j, $j = 1, 2$, we recall some known results for the following system:

$$\partial_t^2 v_j - \Delta v_j + m_j^2 v_j = F_j(V, \partial_t V, \partial_x V), \quad j = 1, 2, \tag{3}$$

on $\mathbb{R}^d \times \mathbb{R}$, with given smooth, compact supported Cauchy data of size $\epsilon \ll 1$, where $V = (v_1, v_2)$, and where nonlinearities F_j are at least of order 2 at the origin. In high dimensions, say, $d \geq 3$, the methods of Klainermann [31] and Shatah [41] are available for the system, so one gets global existence in these dimensions. The cases of $d = 2$ and $d = 1$ are more delicate as there is a similar difficulty as in the case of a single equation. Nevertheless, the authors in [17] are able to deal with the case $d = 2$. They obtained a global solution under the assumption that either the system is non resonant (i.e., $m_1 \neq 2m_2$ and $m_2 \neq 2m_1$) or it is resonant (i.e., $m_1 = 2m_2$) and nonlinearities satisfy some structure condition (see also [43] for the non resonance case). When $d = 1$, the global existence was proved by Sunagawa [43] under the non resonance assumption (when $d = 1$, non resonance condition reads $(3m_1 - m_2)(m_1 - m_2)(m_1 - 3m_2) \neq 0$). Some other sufficient conditions for the global existence of the system in dimension one were obtained in [22, 48, 47]. For the equation with weakly decaying Cauchy data, the results in [11, 16] still hold. One may also refer to [20, 21, 34] for some results of a system of Klein-Gordon equations with different speed.

Note that in most works we mentioned above, the dispersive property of the linear equation was used. So the following question appears naturally: what can one get when there is no dispersion for the linear equation? The equation (2) on $M \times \mathbb{R}$ with M a compact Riemannian manifold provides such a framework. In fact, if λ^2 is an eigenvalue of the Laplace-Beltrami operator $-\Delta$ on the compact Riemannian manifold (M, g) and $\phi(x)$ is an eigenfunction associated to λ^2, then $v(x,t) = e^{i\sqrt{\lambda^2+m^2}t}\phi(x)$ is a solution of the corresponding linear equation. We see that $v(x,t)$ displays no dispersion but it is periodic instead. Though the linear equation displays no dispersion, it has been shown by Bourgain [6], Bambusi [1], Bambusi et al. [5] that when $M = \mathbb{S}^1$, the solution exists almost globally: for any natural number N, if the data are in $H^{s+1}(\mathbb{S}^1) \times H^s(\mathbb{S}^1)$ for some s depending on N, if m stays outside an exceptional subset of zero measure, the solution exists at least on an interval of length $C_N\epsilon^{-N}$. It has also been showed by Delort et al. [18] that the equation (2) on the sphere $\mathbb{S}^{d-1}$ ($d \geq 2$) with smooth Cauchy data of size

$\epsilon \ll 1$, has a solution defined on an interval of time of length $c\epsilon^{\kappa'}$ with $\kappa' > \kappa$, for almost every $m > 0$. This is a better lower bound of lifespan of the solution than the one given by local existence theory. The key point in the proof is that the spectrum of the operator $\sqrt{-\Delta}$ on the sphere is made of multiple eigenvalues asymptotically close — up to a fixed translation — to the integers, which means that the gap between two successive eigenvalues is bounded from below by a fixed constant. Similar results have been obtained in the case of any Zoll manifold (i.e. any compact Riemannian manifold all of whose geodesics are periodic with the same period) by Delort et al. [19] and later improved by Bambusi et al. [4], where the authors showed almost global existence.

The gap condition plays an essential role in works we mentioned in the above paragraph. A natural question is to examine which lower bounds on the time of existence of solutions might be obtained when the eigenvalues of the operator do not satisfy such a gap condition. The problem has been addressed for the equation $(\partial_t^2 - \Delta + m^2)v = v^{\kappa+1}$ on the torus $\mathbb{T}^d$ when $d \geq 2$ by Delort [14]. It has been proved that for almost every $m > 0$, the solution of such an equation exists over an interval of time of length bounded from below by $c\epsilon^{-\kappa(1+2/d)}$ up to a logarithm and has Sobolev norms of high index bounded on such an interval. Note that two successive eigenvalues λ, λ' of $\sqrt{-\Delta}$ on $\mathbb{T}^d$ might be separated by an interval of length as small as c/λ, which means that the gap shrinks to zero as eigenvalues tend to infinity.

Now let us look at the Cauchy problem (1) on the whole space $\mathbb{R}^d$. The situation is dramatically different from the case when there is no quadratic potential. Since the harmonic oscillator on $\mathbb{R}^d$ has pure point spectrum (see [44]), there is no dispersion effect for the corresponding linear equation. Because of that, the question of long-time existence for Klein-Gordon equations associated to the harmonic oscillator is similar to the corresponding problem on compact manifolds. Moreover, the spectrum of the harmonic oscillator on $\mathbb{R}^d$ has a similar gap condition as in the case of $\sqrt{-\Delta}$ on $\mathbb{T}^d$, namely, two successive eigenvalues λ, λ' of $\sqrt{-\Delta + |x|^2}$ on $\mathbb{R}^d$ might be separated by an interval of length as small as c/λ. An interesting question is what we may get for the model related to the harmonic oscillator. In this thesis, we exploit this gap condition to get for the corresponding equation a lower bounded of the time of existence of order $c\epsilon^{-4\kappa/3}$ when $d \geq 2$ (and a slightly better bound if $d = 1$). Note that the estimate we get for the time of existence is explicit (given by the exponent $-4\kappa/3$) and independent of the dimension d. This is contrast with the case of the torus (see Delort [14]), where the gain $2/d$ on the exponent brought by the method goes to zero as $d \to +\infty$. The point is that when the dimension increases, the multiplicity of the eigenvalues of $-\Delta + |x|^2$ grows, while the spacing between different eigenvalues remains essentially the same. We also mention that very recently, Grébert et al. [25] have studied the non-linear Schrödinger equation associated to the harmonic oscillator. They have obtained almost global existence of small solutions for this equation.

We have just said that the problem related to the harmonic oscillator on the whole space is similar to the one on the torus, and the gain on the exponent in the former case is independent of the dimension d. A natural question appears: could we get a lower bound of the lifespan of solutions of order $c\epsilon^{-\kappa(1+\alpha)}$ with $\alpha > 0$ independent of d in the latter case? The second section Chapter 1 is devoted to study this question. Indeed, we shall prove that α can be taken to be a constant as close to $1/2$ as we want. When dimension $d > 4$, this result is better than that of Delort [14].

We use the same method to study the problem of long-time existence of (1) and that of $(\partial_t^2 - \Delta + m^2)v = v^{\kappa+1}$ on the torus. It is based on normal form methods. Such an idea has been first introduced in the study of non-linear Klein-Gordon equation on $\mathbb{R}^d$ by Shatah [41] and is at the root of the results obtained on $\mathbb{S}^1, \mathbb{S}^d, \mathbb{T}^d$ in [6, 1, 5, 4, 14]. In particular, we do not need to use any KAM results, unlike in the study of periodic or quasi-periodic solutions of semi-linear wave or Klein-Gordon equations. For such a line of studies, we refer to the books of Kuksin [35, 36] and Craig [10] in the case of the equation on $\mathbb{S}^1$, to Berti et al. [3] for recent results on the sphere, and to Bourgain [9] and Elliasson et al. [24] in the case of the torus.

Let us explain the idea briefly on the model $(\partial_t^2 - \Delta + m^2)v = v^{\kappa+1}$ on the torus $\mathbb{T}^d$. The goal is to control the Sobolev energy computing

$$\frac{d}{dt}[\|v(\cdot,t)\|^2_{H^{s+1}} + \|\partial_t v(\cdot,t)\|^2_{H^s}]. \tag{4}$$

Using the equation, we may write this quantity as a multilinear expression in $v, \partial_t v$ homogeneous of degree $\kappa + 2$. We then perturb the Sobolev energy by an expression homogeneous of degree $\kappa + 2$ so that its time derivative cancels out the main contribution in (4), up to a remainder of higher order. Moreover, we require that the perturbation we construct can be controlled by powers of $\|v(\cdot,t)\|_{H^{s+1}} + \|\partial_t v(\cdot,t)\|_{H^s}$, with the same s as in (4). Using an expansion of elements of H^s on a basis of L^2 made of eigenfunctions of $\sqrt{-\Delta}$, we are reduce to study expressions of type

$$\sum_{n_1,\dots,n_{\kappa+1}\in\mathbb{N}} F_m(\lambda_{n_0},\dots,\lambda_{n_{\kappa+1}})^{-1} \int_{\mathbb{T}^d} (\Pi_{\lambda_{n_0}} u_0)\dots(\Pi_{\lambda_{n_{\kappa+1}}} u_{\kappa+1})(\lambda_{n_0}+\dots+\lambda_{n_{\kappa+1}})^{2s} dx, \tag{5}$$

where $\{\lambda_n\}_{n\in\mathbb{N}}$ is the spectrum of $\sqrt{-\Delta}$ on $\mathbb{T}^d$, Π_λ is the spectral projector associated to the eigenvalue λ and F_m is given by

$$F_m(\xi_0,\dots,\xi_{\kappa+1}) = \sum_{j=0}^{\kappa+1} e_j\sqrt{m^2+\xi_j^2}, \quad e_j \in \{-1,1\}. \tag{6}$$

The problem is to bound $|F_m(\lambda_{n_0},\dots,\lambda_{n_{\kappa+1}})|$ from below, for those λ_{n_j} for which (6) is non zero, in such a way that (5) can be bounded from above by $C\prod\|u_j\|_{H^s}$ for s large enough. It has been proved that if for almost every $m > 0$, there are $c > 0$, $N_0 \in \mathbb{N}$ such that

$$|F_m(\lambda_{n_0},\dots,\lambda_{n_{\kappa+1}})| \geq c(1 + \text{ the third largest among } \{\lambda_{n_0},\dots,\lambda_{n_{\kappa+1}}\})^{-N_0} \tag{7}$$

holds for all those $\lambda_{n_0}, \dots, \lambda_{n_{\kappa+1}}$ for which $F_m(\lambda_{n_0}, \dots, \lambda_{n_{\kappa+1}})$ is non zero, then (5) can be bounded from above by $C \prod \|u_j\|_{H^s}$ (see for instance [18]). This is true in the case of spheres, but it does not holds generally in the case of tori. In fact, in the case of tori we will have the following two cases: the first case is that the two largest elements of $\{e_j \sqrt{m^2 + \lambda_{n_j}^2};\ 0 \leq j \leq \kappa+1\}$ have same sign. In this case, one may obtain that for almost every $m > 0$, there are $c > 0$, $N_0 \in \mathbb{N}$ such that

$$\begin{aligned} |F_m(\lambda_{n_0}, \dots, \lambda_{n_{\kappa+1}})| \geq c(1 + \{\lambda_{n_0}, \dots, \lambda_{n_{\kappa+1}}\}^1 + \{\lambda_{n_0}, \dots, \lambda_{n_{\kappa+1}}\}^2) \\ \times (1 + \text{ the third largest among } \{\lambda_{n_0}, \dots, \lambda_{n_{\kappa+1}}\})^{-N_0} \end{aligned} \tag{8}$$

holds for those $\lambda_{n_0}, \dots, \lambda_{n_{\kappa+1}}$ for which $F_m(\lambda_{n_0}, \dots, \lambda_{n_{\kappa+1}}) \neq 0$, where $\{\lambda_{n_0}, \dots, \lambda_{n_{\kappa+1}}\}^i$ stands for the i-th largest element among $\{\lambda_{n_0}, \dots, \lambda_{n_{\kappa+1}}\}$, $i = 1, 2$. Plugging into (5), we see that the loss is given by a large power of small frequencies (indeed we gain one power of the largest frequency), which allows us to control (5) by $C \prod_j \|u_j\|_{H^s}$ for large enough s. The second case is that the two largest elements of $\{e_j \sqrt{m^2 + \lambda_{n_j}^2};\ 0 \leq j \leq \kappa+1\}$ have opposite signs. In this case, we are only able to show, using harmonic analysis on $\mathbb{T}^d$, that for any $\rho > 0$, for almost every $m > 0$, there are $c > 0$, $N_0 \in \mathbb{N}$ such that

$$\begin{aligned} |F_m(\lambda_{n_0}, \dots, \lambda_{n_{\kappa+1}})| \geq c(1 + \{\lambda_{n_0}, \dots, \lambda_{n_{\kappa+1}}\}^1 + \{\lambda_{n_0}, \dots, \lambda_{n_{\kappa+1}}\}^2)^{-3-\rho} \\ \times (1 + \text{ the third largest among } \{\lambda_{n_0}, \dots, \lambda_{n_{\kappa+1}}\})^{-N_0} \end{aligned} \tag{9}$$

holds for those $\lambda_{n_0}, \dots, \lambda_{n_{\kappa+1}}$ for which $F_m(\lambda_{n_0}, \dots, \lambda_{n_{\kappa+1}}) \neq 0$. When plugging (9) into (5), we see that there is a loss of $3+\rho$ derivatives of high frequencies, besides a loss of a power of low frequencies which is harmless. However, solving the equation makes gain one derivative since the nonlinearity involves no derivative of v and we may gain one more derivative through commutators. This allows us to recover the loss and get an upper bound of (5) by an expression of type $\prod \|u_j\|_{H^s}$ through partition of frequencies between zones $\{\lambda_{n_j} \leq \epsilon^{-\kappa\theta},\ j = 1, \dots, \kappa+1\}$ and $\{\lambda_{n_j} > \epsilon^{-\kappa\theta}$ for at least one $j \in \{1, \dots, \kappa+1\}\}$, where θ is a constant. On the other hand, by the Hamiltonian structure and commutator, we may gain two derivatives for the remainder part of the nonlinearity which we do not use a normal form to cancel out. Thus we can show that the Sobolev energy is bounded on some interval of time and the solution exists on this interval.

Let us give some indication about the proof of (9). For the convenience of expression, we assume $\lambda_{n_0}, \lambda_{n_{\kappa+1}} \gg \lambda_{n_0} + \dots + \lambda_\kappa$. We first notice that one only needs to show that, for any compact interval $I \subset (0, +\infty)$, the measure of the set

$$\{m \in I; |F_m(\lambda_{n_0}, \dots, \lambda_{n_{\kappa+1}})| < r\}$$

where r is the right hand side of (9) with c replaced by α, goes to zero as α tends to zero. Using tools of subanalytic geometry, the interval I may be written for any fixed $n_0, \dots, n_{\kappa+1}$ as the union of a uniform number of intervals on which $|\partial F_m / \partial m|$ can be bounded from below by a large negative power of small frequencies $(1 + \lambda_{n_1} + \dots + \lambda_{n_\kappa})$, and of a remaining set. On each of these intervals, since we have $|\partial F_m/\partial m| \geq C(1+\lambda_{n_1}+\cdots+\lambda_{n_\kappa})^{-N_1}$, we then take F_m as a coordinate so that we can estimate the measure of this interval by $Cr(1+\lambda_{n_1}+\cdots+\lambda_{n_\kappa})^{N_1}$. Taking the expression of r into account, we get an upper bound of the sum of these quantities in $n_0,\ldots,n_{\kappa+1}$ by a constant which goes to zero as α tends to zero. Also using tools of subanalytic geometry we can show that the measure of the remaining set, on which we have $|\partial F_m/\partial m| = O(1+\lambda_{n_1}+\cdots+\lambda_{n_\kappa})^{-N_1}$, is small and goes to zero as α tends to zero. This shows that (9) holds true for those $(n_0,\ldots,n_{p+1}) \in \mathbb{N}^{p+2}$ for which $F_m(\lambda_{n_0},\ldots,\lambda_{n_{\kappa+1}}) \neq 0$ when m is outside a subset of zero measure in I.

Here are some natural open questions that remain unsolved. There is no reason to believe that our results obtained in the first chapter are optimal, so the questions are: we get a lower bound of the time of existence for the Hamiltonian nonlinearities in the thesis; could they be improved? What could we get for nonlinearities involving derivatives of the unknown function deriving or not from a Hamiltonian structure? Note that we used the Hamiltonian structure of the equation to gain one derivative.

In the second chapter of this thesis, we consider the time dependent linear Schrödinger equations:

$$i\partial_t u - \Delta u + V(x,t)u = 0 \tag{10}$$

on $\mathbb{T}^d \times \mathbb{R}$. We want to find a upper bound for the Sobolev norm of the solution. The problem of finding optimal bounds for $||u(t,\cdot)||_{H^s}$ has been addressed by Nenciu [39] and Barbaroux and Joye [2], in the abstract framework of an operator P (instead of $-\Delta$) and a perturbation $V(t)$ acting on elements of a Hilbert space, when the spectrum of P is discrete and has increasing gaps. This condition is satisfied by the Laplacian on the circle. It follows from the results of [39, 2], that solutions of (10) verify

$$||u(t,\cdot)||_{H^s} \leq C_\epsilon |t|^\epsilon ||u(0,\cdot)||_{H^s} \tag{11}$$

when t goes to infinity, for any $\epsilon > 0$. Later, Bourgain [8] proved that a similar bound holds for solutions of (10) on the torus $\mathbb{T}^d$. The increasing gap condition of Nenciu and Barbaroux-Joye is no longer satisfied, and has to be replaced by a convenient decomposition of $\mathbb{Z}^d$ in well separated clusters. Delort [15] recently published a simpler proof of the results of Bourgain (included for other examples of compact manifolds than the torus), which is close to the one of Nenciu and Barbaroux-Joye. If one further assumes that V is analytic, and quasi-periodic in t, then it was showed by Bourgain [7] that (11) holds with $(1+|t|)^\epsilon$ replaced by some power of $\log t$ when $t > 2$. When the dimension $d = 1$, for any real analytic potential, whose holomorphic extension to $\Omega_{\tilde\rho}$ is bounded, where, for some $\tilde\rho > 0$,

$$\Omega_{\tilde\rho} = \{(x,t) \in \mathbb{C}\times\mathbb{C} \,:\, |\operatorname{Im} x| < \tilde\rho,\ |\operatorname{Im} t| < \tilde\rho\},$$

Wang [45] showed that one may still obtain such a logarithmic bound, using the method of [8]. In this paper, we improve the method of Delort [15] to provide a new proof of the result of Wang [45] and extend it to any dimension $d \geq 1$ and to Gevrey regularity. Our result is the following: if $V(x,t)$ is a real smooth function on $\mathbb{T}^d \times \mathbb{T}$ and it is a Gevrey-μ function in time t and Gevrey-ν in every space variable, then there is $\zeta > 0$ independent of μ and ν such that for any $s > 0$, there is a constant $C_{s,\nu,d} > 0$ such that

$$||u(t)||_{H^s} \leq C_{s,\nu,d}\big[\log(2+|t|)\big]^{\zeta\mu\nu s}||u(0)||_{H^s}, \tag{12}$$

where $u(t)$ is the solution to (10) with the initial condition $u(0) \in H^s(\mathbb{T}^d)$.

There are also some results about uniformly bounded Sobolev norms. Eliasson and Kuksin [23] have shown that if the potential V on $\mathbb{T}^d \times \mathbb{R}$ is analytic in space, quasi-periodic in time, and small enough, then for most values of the parameter of quasi-periodicity, the equation reduces to an autonomous one. Consequently, the Sobolev norm of the solution is uniformly bounded. A similar result for the harmonic oscillator has been obtained by Grébert and Thomann [27] recently. For Schrödinger equations on the circle with a small time periodic potential, Wang [46] showed that the solutions of the corresponding equation have bounded Sobolev norms.

Now let us give a picture of the proof of our result. For any given $N \in \mathbb{N}^*$, one first finds for every fixed time t an operator $Q^N(\cdot,t)$, which extends as a bounded linear operator from $H^N(\mathbb{T}^d)$ to $H^N(\mathbb{T}^d)$ such that

$$\big(I + Q^N(\cdot,t)\big)^*(i\partial_t - \Delta + V)\big(I + Q^N(\cdot,t)\big) = i\partial_t - \Delta + V'_N(\cdot,t) + R'_N(\cdot,t) \tag{13}$$

with self-adjoint operator V'_N exactly commuting to the modified Laplacian $\tilde{\Delta}$ (see Chapter 2 for its precise definition) and R'_N a remainder operator which is essentially a bounded linear map from $L^2(\mathbb{T}^d)$ to $H^N(\mathbb{T}^d)$. Moreover, we also require that the adjoint of Q^N in the usual L^2 paring (denoted by $Q^N(\cdot,t)^*$) extends as a bounded linear operator from $H^N(\mathbb{T}^d)$ to $H^N(\mathbb{T}^d)$. In order to obtain the estimate for the solution u of (10), one needs to 'invert' the operator $I + Q^N$, that is , to find an operator P^N, which extends as a bounded linear operator not only from $H^N(\mathbb{T}^d)$ to $H^N(\mathbb{T}^d)$, but also from $L^2(\mathbb{T}^d)$ to $L^2(\mathbb{T}^d)$, such that

$$\big(I + Q^N(\cdot,t)\big)\big(I + P^N(\cdot,t)\big) = I + R_N(\cdot,t) \tag{14}$$

where R_N is a remainder operator such that $[i\partial_t - \Delta + V, R_N]$ sends $L^2(\mathbb{T}^d)$ to $H^N(\mathbb{T}^d)$. Now by setting

$$v = (I + P^N)u, \tag{15}$$

we deduce from (13), (14) and (10)

$$(i\partial_t - \Delta + V'_N)v = (I + Q^N)^*[i\partial_t - \Delta + V, R_N]u - R'_N v. \tag{16}$$

Remarking that the modified Laplacian has the property that

$$C^{-N}\|(1-\Delta)^{\frac{N}{2}}u\|_{L^2} \leq \|(1-\tilde{\Delta})^{\frac{N}{2}}u\|_{L^2} \leq C^N\|(1-\Delta)^{\frac{N}{2}}u\|_{L^2}$$

holds for any $u \in H^N(\mathbb{T}^d)$ and for some uniform constant C, then we let the operator $(1-\tilde{\Delta})^{\frac{N}{2}}$ act on both sides of (16) and deduce from the energy inequality

$$||v(t)||_{H^N} \leq C_N ||v(0)||_{H^N} + C_N \int_0^t ||(I+Q^N)^*[i\partial_t - \Delta + V, R_N]u(t)||_{H^N} + ||R'_N v(t)||_{H^N} dt,$$

which together with (15), the conservation law of the L^2-norm of (10) and the properties of those operators we have constructed, implies

$$||v(t)||_{H^N} \leq C_N ||v(0)||_{H^N} + C_N |t| ||u(0)||_{L^2}. \tag{17}$$

We then use (14), (15) and the properties of the operators to deduce

$$||u(t)||_{H^N} \leq C_N \Big(||u(0)||_{H^N} + (2+|t|)||u(0)||_{L^2} \Big). \tag{18}$$

Remark that the above constants C_N may be different in different lines and they depend on the norms of operators which appear in the above process. Since (18) holds for any $N \in \mathbb{N}^*$, if we have good estimates for C_N (we shall finally see that C_N can be controlled by C^N times a power of the factorial of N), then the theorem will follow by interpolation between (18) with $N=0$ and some N much larger than s. There are two difficulties. The first one is that we have to carefully choose those operators Q^N so that the above process can go on. The second is to obtain proper estimates for C_N, which means that we have to estimate the norms of operators and remainders for every $N \in \mathbb{N}^*$ in the above process. The proof of such estimates is the main difficulty of the paper. We follow the construction method above, keeping track at each step of the constants, and exploiting the assumptions of Gevrey-regularity in space and time made on the potential.

Remark *The notation systems are independent in each section.*

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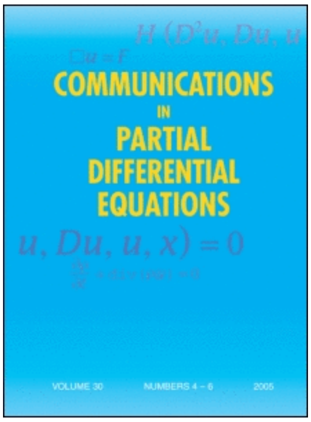
Chapter 1

Long-Time Existence

§1.1

Long-Time Existence for Semi-Linear Klein-Gordon Equations with Quadratic Potential



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Long-Time Existence for Semi-Linear Klein–Gordon Equations with Quadratic Potential

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We prove that small smooth solutions of semi-linear Klein–Gordon equations with quadratic potential exist over a longer interval than the one given by local existence theory, for almost every value of mass. We use normal form for the Sobolev energy. The difficulty in comparison with some similar results on the sphere comes from the fact that two successive eigenvalues λ, λ' *of* $\sqrt{-\Delta+|x|^2}$ *may be separated by a distance as small as* $\frac{1}{\lambda}$.

Keywords Harmonic oscillator; Klein–Gordon; Long-time existence.

Mathematics Subject Classification 35L15; 35H70.

Introduction

Let $-\Delta+|x|^2$ be the harmonic oscillator on $\mathbb{R}^d$. This paper is devoted to the proof of lower bounds for the existence time of solutions of non-linear Klein–Gordon equations of type

$$\begin{aligned}(\partial_t^2-\Delta+|x|^2+m^2)v &= v^{\kappa+1}\\ v|_{t=0} &= \epsilon v_0\\ \partial_t v|_{t=0} &= \epsilon v_1\end{aligned}$$

where $m\in\mathbb{R}_+^*$ (we denote $(0,+\infty)$ by R_+^* throughout the paper), $x^\alpha\partial_x^\beta v_j\in L^2$ when $|\alpha|+|\beta|\le s+1-j$ $(j=0,1)$ for a large enough integer s, and where $\epsilon>0$ is small enough.

The similar equation without the quadratic potential $|x|^2$, and with data small, smooth and compactly supported, has almost global solutions when $d=1$ (see Moriyama et al. [22]), and has global solutions when $d\ge 2$ (see Klainerman [18]

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and Shatah [24] for dimensions $d \geq 3$, Ozawa et al. [23] and Delort et al. [10] when $d = 2$). The situation is drastically different when we replace $-\Delta$ by $-\Delta + |x|^2$, since the latter operator has pure point spectrum. This prevents any time decay for solutions of the linear equation. Because of that, the question of long time existence for Klein–Gordon equations associated to the harmonic oscillator is similar to the corresponding problem on compact manifolds. One may refer to [12] for the corresponding problem on Zoll manifolds.

For the equation $(\partial_t^2 - \Delta + m^2)v = v^{\kappa+1}$ on the circle $\mathbb{S}^1$, it has been proved by Bourgain [6] and Bambusi [1], that for almost every $m > 0$, the above equation has solutions defined on intervals of length $c_N\epsilon^{-N}$ for any $N \in \mathbb{N}$, if the data are smooth and small enough (see also the lectures of Grébert [14]). These results have been extended to the sphere $\mathbb{S}^d$ instead of $\mathbb{S}^1$ by Bambusi et al. [2]. A key property in the proofs is the structure of the spectrum of $\sqrt{-\Delta}$ on $\mathbb{S}^d$. It is made of the integers, up to a small perturbation, so that the gap between two successive eigenvalues is bounded from below by a fixed constant.

A natural question is to examine which lower bounds on the time of existence of solutions might be obtained when the eigenvalues of the operator do not satisfy such a gap condition. The problem has been addressed for $(\partial_t^2 - \Delta + m^2)v = v^{\kappa+1}$ on the torus $\mathbb{T}^d$ when $d \geq 2$ by Delort [9]. It has been proved that for almost every $m > 0$, the solution of such an equation exists over an interval of time of length bounded from below by $c\epsilon^{-\kappa(1+2/d)}$ (up to a logarithm) and has Sobolev norms of high index bounded on such an interval. Note that two successive eigenvalues λ, λ' of $\sqrt{-\Delta}$ on $\mathbb{T}^d$ might be separated by an interval of length as small as c/λ. One may naturally ask what may happen in the case of the harmonic oscillator since the eigenvalues of $\sqrt{-\Delta + |x|^2}$ on $\mathbb{R}^d$ share the similar gap condition as in the case of the torus. Our goal is to exploit this fact to get for the corresponding Klein–Gordon equation a lower bound of the time of existence of order $c\epsilon^{-4\kappa/3}$ when $d \geq 2$ (and a slightly better bound if $d = 1$).

Note that the estimate we get for the time of existence is explicit (given by the exponent $-4\kappa/3$) and independent of the dimension d. This is in contrast with the case of the torus, where the gain $2/d$ on the exponent brought by the method goes to zero as $d \to +\infty$. The point is that when the dimension increases, the multiplicity of the eigenvalues of $-\Delta + |x|^2$ grows, while the spacing between different eigenvalues remains essentially the same.

The method we use is based, as for similar problems on the sphere and the torus, on normal form methods. Such an idea has been introduced in the study of non-linear Klein–Gordon equations on $\mathbb{R}^d$ by Shatah [24], and is at the root of the results obtained on $\mathbb{S}^1, \mathbb{S}^d, \mathbb{T}^d$ in [1–3, 6, 9]. In particular, we do not need to use any KAM results, unlike in the study of periodic or quasi-periodic solutions of semi-linear wave or Klein–Gordon equations. For such a line of studies, we refer to the books of Kuksin [20, 21] and Craig [8] in the case of the equation on $\mathbb{S}^1$, to Berti and Bolle [4] for recent results on the sphere, and to Bourgain [7] and Elliasson and Kuksin [13] in the case of the torus.

Finally let us mention that very recently Grébert et al. [15] have studied the non-linear Schrödinger equation associated to the harmonic oscillator. They have obtained almost global existence of small solutions for this equation.

1. The Semi-Linear Klein–Gordon Equation

1.1. *Sobolev Spaces*

We introduce in this subsection Sobolev spaces we will work with. From now on, we denote by $P = \sqrt{-\Delta + |x|^2}$, $x \in \mathbb{R}^d$, $d \geq 1$. The operator $P^2 = -\Delta + |x|^2$ is called the harmonic oscillator on $\mathbb{R}^d$. The eigenvalues of P^2 are given by λ_n^2, where

$$\lambda_n = \sqrt{2n + d}, \quad n \in \mathbb{N}. \tag{1.1.1}$$

Let Π_n be the orthogonal projector to the eigenspace associated to λ_n^2. There are several ways to characterize these spaces. Of course we will show they are equivalent after giving definitions.

Definition 1.1.1. Let $s \in \mathbb{R}$. We define $\mathscr{H}_1^s(\mathbb{R}^d)$ to be the set of all functions $u \in L^2(\mathbb{R}^d)$ such that $(\lambda_n^s \|\Pi_n u\|_{L^2})_{n\in\mathbb{N}} \in \ell^2$, equipped with the norm defined by $\|u\|_{\mathscr{H}_1^s}^2 = \sum_{n\in\mathbb{N}} \lambda_n^{2s} \|\Pi_n u\|_{L^2}^2$.

The space $\mathscr{H}_1^s(\mathbb{R}^d)$ is the domain of the operator $g(P)$ on $L^2(\mathbb{R}^d)$, which is defined using functional calculus and where

$$g(r) = (1 + r^2)^{\frac{s}{2}}, \quad r \in \mathbb{R}. \tag{1.1.2}$$

Because of (1.1.1), we have

$$\|g(P)u\|_{L^2} \sim \|u\|_{\mathscr{H}_1^s}. \tag{1.1.3}$$

Definition 1.1.2. Let $s \in \mathbb{N}$. We define $\mathscr{H}_2^s(\mathbb{R}^d)$ to be the set of all functions $u \in L^2(\mathbb{R}^d)$ such that $x^\alpha \partial^\beta u \in L^2(\mathbb{R}^d)$, $\forall |\alpha| + |\beta| \leq s$, equipped with the norm defined by $\|u\|_{\mathscr{H}_2^s}^2 = \sum_{|\alpha|+|\beta|\leq s} \|x^\alpha \partial^\beta u\|_{L^2}^2$.

We shall give another definition of the space in the view point of pseudo-differential theory. Let us first list some results from [16].

Definition 1.1.3. We denote by $\Gamma^s(\mathbb{R}^d)$, where $s \in \mathbb{R}$, the set of all functions $a \in C^\infty(\mathbb{R}^d)$ such that: $\forall \alpha \in \mathbb{N}^d$, $\exists\ C_\alpha$, s.t. $\forall z \in \mathbb{R}^d$, we have $|\partial_z^\alpha a(z)| \leq C_\alpha \langle z \rangle^{s-|\alpha|}$, where $\langle z \rangle = (1 + |z|^2)^{\frac{1}{2}}$.

Definition 1.1.4. Assume $a_j \in \Gamma^{s_j}(\mathbb{R}^d)$ ($j \in \mathbb{N}^*$) and that s_j is a decreasing sequence tending to $-\infty$. We say a function $a \in C^\infty(\mathbb{R}^d)$ satisfies:

$$a \sim \sum_{j=1}^{\infty} a_j$$

if: $\forall r \geq 2$, $r \in \mathbb{N}$, $a - \sum_{j=1}^{r-1} a_j \in \Gamma^{s_r}(\mathbb{R}^d)$.

We now would like to consider operators of the form

$$Au(x) = (2\pi)^{-d} \iint e^{i(x-y)\cdot\xi} a(x, \xi) u(y) dy\, d\xi \tag{1.1.4}$$

where $a(x, \xi) \in \Gamma^s(\mathbb{R}^{2d})$. We can also consider a more general formula for the action of the operator

$$Au(x) = (2\pi)^{-d} \iint e^{i(x-y)\cdot\xi} a(x, y, \xi)u(y)dy\, d\xi \tag{1.1.5}$$

where the function $a(x, y, \xi)$ is called the amplitude. We will describe the class of amplitudes as the following:

Definition 1.1.5. Let $s \in \mathbb{R}$ and $\Omega^s(\mathbb{R}^{3d})$ denote the set of functions $a(x, y, \xi) \in C^\infty(\mathbb{R}^{3d})$, which for some $s' \in \mathbb{R}$ satisfy

$$|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(x, y, \xi)| \le C_{\alpha\beta\gamma} \langle z \rangle^{s-(|\alpha|+|\beta|+|\gamma|)} \langle x - y \rangle^{s'+|\alpha|+|\beta|+|\gamma|},$$

where $z = (x, y, \xi) \in \mathbb{R}^{3d}$.

The following proposition is a special case of Proposition 1.1.4 in [16].

Proposition 1.1.6. *If* $b \in \Gamma^s(\mathbb{R}^{2d})$, *then* $a(x, y, \xi) = b(x, \xi)$ *and* $a(x, y, \xi) = b(y, \xi)$ *belong to* $\Omega^s(\mathbb{R}^{3d})$.

Let $\chi(x, y, \xi) \in C_0^\infty(\mathbb{R}^{3d})$, $\chi(0, 0, 0) = 1$. It is shown by Lemma 1.2.1 in [16] that (1.1.5) makes sense in the following way:

$$Au(x) = \lim_{\varepsilon \to +0} (2\pi)^{-d} \iint e^{i(x-y)\cdot\xi} \chi(\varepsilon x, \varepsilon y, \varepsilon\xi) a(x, y, \xi)u(y)dy\, d\xi \tag{1.1.6}$$

if $a(x, y, \xi) \in \Omega^s(\mathbb{R}^{3d})$ for some s. It is also shown in the same section of it the operator A is continuous from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}(\mathbb{R}^d)$ and it can be uniquely extended to an operator from $\mathscr{S}'(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$.

Definition 1.1.7. The class of pseudo-differential operators A of the form (1.1.5) with amplitudes $a \in \Omega^s(\mathbb{R}^{3d})$ will be denoted by $G^s(\mathbb{R}^d)$.

We set $G^{-\infty}(\mathbb{R}^d) = \bigcap_{s\in\mathbb{R}} G^s(\mathbb{R}^d)$.

Example 1.1.8. For $s \in \mathbb{N}$, the constant coefficient differential operator $\sum_{|\alpha|+|\beta|\le s} c_{\alpha\beta} x^\alpha \partial^\beta$ is in the class $G^s(\mathbb{R}^d)$.

The class $G^s(\mathbb{R}^d)$ has some properties which are just Theorems 1.3.1, 1.4.7, 1.4.8 in [16].

Theorem 1.1.9. *Let* $s_1, s_2 \in \mathbb{R}$ *and* $A \in G^{s_1}(\mathbb{R}^d)$, $A' \in G^{s_2}(\mathbb{R}^d)$. *Then* $A \circ A' \in G^{s_1+s_2}(\mathbb{R}^d)$.

Theorem 1.1.10. *The operator* $A \in G^0(\mathbb{R}^d)$ *can be extended to a bounded operator on* $L^2(\mathbb{R}^d)$.

Theorem 1.1.11. *The operator* $A \in G^s(\mathbb{R}^d)$ *for* $s < 0$ *can be extended to a compact operator on* $L^2(\mathbb{R}^d)$.

We shall give a subclass of that of pseudo-differential operators.

Definition 1.1.12. We say $a \in \Gamma^s_{cl}(\mathbb{R}^d)$ if $a \in \Gamma^s(\mathbb{R}^d)$ and a has asymptotic expansion:

$$a \sim \sum_{j \in \mathbb{N}} a_{s-j}$$

with $a_{s-j} \in C^\infty(\mathbb{R}^d)$ satisfying for $\theta \geq 1$, $|x| + |\xi| \geq 1$

$$a_{s-j}(\theta x, \theta \xi) = \theta^{s-j} a_{s-j}(x, \xi).$$

Definition 1.1.13. Let A be a pseudo-differential operator with amplitude $a \in \Gamma^s_{cl}(\mathbb{R}^d)$. We then call a_s defined above the principle symbol of A.

Definition 1.1.14. We say a pseudo-differential operator $A \in G^s_{cl}(\mathbb{R}^d)$ if its amplitude $a \in \Gamma^s_{cl}(\mathbb{R}^{2d})$.

By Proposition 1.1.6, Definition 1.1.14 is meaningful.

Definition 1.1.15. We say that $A \in G^s_{cl}(\mathbb{R}^d)$ is globally elliptic if we have: $\exists R > 0$, $\exists C > 0$ such that $\forall (x, \xi) \in \mathbb{R}^{2d}$ satisfying $|x| + |\xi| \geq R$, we have $|a_s(x, \xi)| \geq C(|x| + |\xi|)^s$, where a_s denotes the principle symbol of A.

We can invert the operator $A \in G^s_{cl}(\mathbb{R}^d)$ up to a regularizing operator, which is just Theorem 1.5.7 in [16].

Theorem 1.1.16. *Let $A \in G^s_{cl}(\mathbb{R}^d)$ be a globally elliptic operator. Then there is an operator $B \in G^{-s}_{cl}(\mathbb{R}^d)$ such that*

$$B \circ A = I + R_1, \quad A \circ B = I + R_2, \tag{1.1.7}$$

where R_1, R_2 are regularizing, i.e., $R_1, R_2 \in G^{-\infty}(\mathbb{R}^d)$.

Definition 1.1.17. Let $s \in \mathbb{R}$ and A a pseudo-differential operator whose symbol is $\langle \xi, x \rangle^s$ modulo Γ^{s-1}_{cl}. We define $\mathcal{H}^s_3(\mathbb{R}^d)$ to be the set of all functions $u \in \mathcal{S}'(\mathbb{R}^d)$ such that $Au \in L^2(\mathbb{R}^d)$, equipped with the norm defined by $\|u\|^2_{\mathcal{H}^s_3} = \|Au\|^2_{L^2} + \|u\|^2_{L^2}$.

Remark 1.1.1. The pseudo-differential operator A defined above is globally elliptic. Thus by Theorem 1.1.16 if $Au \in L^2(\mathbb{R}^d)$, we must have $u \in L^2(\mathbb{R}^d)$.

Remark 1.1.2. $\mathcal{H}^s_3(\mathbb{R}^d)$ does not depend on the choice of A according to Corollary 1.6.5 in [16].

Corollary 1.1.18. *When $s \in \mathbb{N}$, Definitions* 1.1.1, 1.1.2 *and* 1.1.17 *characterize the same space. Moreover $\mathcal{H}^s_3(\mathbb{R}^d) = \mathcal{H}^s_1(\mathbb{R}^d)$ for any $s \in \mathbb{R}$.*

Proof. First let $s \in \mathbb{N}$. Since A in Definition 1.1.17 is globally elliptic, by Theorem 1.1.16 there is $B \in G^{-s}_{cl}(\mathbb{R}^d)$ such that

$$B \circ A = I + R_1, \quad A \circ B = I + R_2 \tag{1.1.8}$$

where R_1, R_2 are regularizing. Thus for any α, β with $|\alpha| + |\beta| \leq s$, by the example after Definition 1.1.12 and Theorems 1.1.9–1.1.11, we have $\|x^\alpha \partial^\beta u\|_{L^2} \leq \|x^\alpha \partial^\beta BAu\|_{L^2} + \|x^\alpha \partial^\beta R_1 u\|_{L^2} \leq C(\|Au\|_{L^2} + \|u\|_{L^2})$, which implies $\|u\|_{\mathscr{H}_2^s} \leq C\|u\|_{\mathscr{H}_3^s}$. The inverse inequality follows from the proof of Proposition 1.6.6 in [16]. Let us now prove that Definition 1.1.1 is equivalent to Definition 1.1.17 for any $s \in \mathbb{R}$.

By Theorem 1.11.2 in [16] the operator $g(P)$ defined in (1.1.2) is an essentially self-adjoint globally elliptic operator in the class $G^s(\mathbb{R}^d)$. We have again by Theorem 1.1.16 that there is $Q \in G_{cl}^{-s}(\mathbb{R}^d)$ such that

$$g(P) \circ Q = I + R_1', \quad Q \circ g(P) = I + R_2' \tag{1.1.9}$$

where R_1', R_2' are regularizing. We compute using (1.1.3), (1.1.8), (1.1.9) together with Theorems 1.1.9 and 1.1.10

$$\begin{aligned} \|u\|_{\mathscr{H}_1^s} \sim \|g(P)u\|_{L^2} &\leq \|(g(P) \circ B \circ A)u\|_{L^2} + \|(g(P) \circ R_1)u\|_{L^2} \\ &\leq C(\|Au\|_{L^2} + \|u\|_{L^2}) \leq C\|u\|_{\mathscr{H}_3^s} \end{aligned}$$

and

$$\begin{aligned} \|u\|_{\mathscr{H}_3^s} &\leq C(\|(A \circ Q \circ g(P))u\|_{L^2} + \|(A \circ R_2')u\|_{L^2} + \|u\|_{L^2}) \\ &\leq C(\|g(P)u\|_{L^2} + \|u\|_{L^2}) \leq C\|u\|_{\mathscr{H}_1^s}, \end{aligned}$$

where the last inequality follows from the fact $\lambda_n \geq 1$. □

We denote $\mathscr{H}^s(\mathbb{R}^d) = \mathscr{H}_1^s(\mathbb{R}^d) = \mathscr{H}_3^s(\mathbb{R}^d)$ when $s \in \mathbb{R}$. When $s \in \mathbb{N}$, this space coincides with $\mathscr{H}_2^s(\mathbb{R}^d)$. Let us present some properties of the spaces we shall use.

Proposition 1.1.19. *If $s_1 \leq s_2$, then $\mathscr{H}^{s_2}(\mathbb{R}^d) \hookrightarrow \mathscr{H}^{s_1}(\mathbb{R}^d)$.*

Proposition 1.1.20. *If $s > d/2$, then $\mathscr{H}^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$.*

Proposition 1.1.21. *Let $f \in C^\infty(\mathbb{R})$, $f(0) = 0$, $u \in \mathscr{H}^s(\mathbb{R}^d)$, $s \in \mathbb{N}$, $s > d$. Then we have $f(u) \in \mathscr{H}^s(\mathbb{R}^d)$. Moreover if f vanishes at order $p+1$ at 0, where $p \in \mathbb{N}$, then $\|f(u)\|_{\mathscr{H}^s} \leq C\|u\|_{\mathscr{H}^s}^{p+1}$.*

Proof. Propositions 1.1.19 and 1.1.20 follow respectively from the definition and Sobolev embedding. By the chain rule, for $|\alpha| + |\beta| \leq s$, $x^\alpha \partial^\beta f(u)$ may be written as the sum of terms of following form:

$$x^\alpha f^{(k)}(u)(\partial^{\beta_1} u) \cdots (\partial^{\beta_k} u),$$

where $k \leq s$, $|\alpha| + \sum_{i=1}^k |\beta_i| \leq s$, $|\beta_i| > 0$, $i = 1, \ldots, k$. Let j_0 be the index such that $|\beta_{j_0}|$ is the largest among $|\beta_1|, \ldots, |\beta_k|$. Thus we must have $|\beta_i| \leq \frac{s}{2}$, $i \neq j_0$. By the assumption on s and Proposition 1.1.20, $\partial^\gamma u \in L^\infty(\mathbb{R}^d)$ if $|\gamma| \leq \frac{d}{2}$. We then estimate the factor $x^\alpha \partial^{\beta_{j_0}} u$ of the above quantities in L^2-norm and others in L^∞-norm. Thus we have $f(u) \in \mathscr{H}^s(\mathbb{R}^d)$ by Proposition 1.1.20. When f vanishes at 0 at order $p+1$, by Taylor formula there is a smooth function h such that $f(u) = u^{p+1}h(u)$. Then we argue as above to get an upper bound of $\|f(u)\|_{\mathscr{H}^s}$ by $C\|u\|_{\mathscr{H}^s}^p \|u\|_{\mathscr{H}^s}$. This concludes the proof. □

Remark 1.1.3. Proposition 1.1.21 actually holds true for $s > d/2$ if we argue as the proof of Corollary 6.4.4 in [17]. Since we will consider only in $\mathcal{H}^s(\mathbb{R}^d)$ for large s, the lower bound of s is not important.

1.2. *Statement of Main Theorem*

Let d be an integer, $d \geq 1$ and $F : \mathbb{R} \to \mathbb{R}$ a real valued smooth function vanishing at order $\kappa + 1$ at 0, $\kappa \in \mathbb{N}^*$ (we denote $\mathbb{N}\backslash\{0\}$ by $\mathbb{N}^*$ throughout the paper). Let $m \in \mathbb{R}^*_+$. we consider the solution v of the following Cauchy problem:

$$\begin{cases} (\partial_t^2 - \Delta + |x|^2 + m^2)v = F(v) \quad \text{on } [-T, T] \times \mathbb{R}^d \\ v(0, x) = \epsilon v_0 \\ \partial_t v(0, x) = \epsilon v_1, \end{cases} \tag{1.2.1}$$

where $v_0 \in \mathcal{H}^{s+1}(\mathbb{R}^d), v_1 \in \mathcal{H}^s(\mathbb{R}^d)$, and $\epsilon > 0$ is a small parameter. By local existence theory one knows that if s is large enough and $\epsilon \in (0, 1)$, equation (1.2.1) admits for any (v_0, v_1) in the unit ball of $\mathcal{H}^{s+1}(\mathbb{R}^d) \times \mathcal{H}^s(\mathbb{R}^d)$ a unique smooth solution defined on the interval $|t| \leq c\epsilon^{-\kappa}$, for some uniform positive constant c. Moreover, $\|v(t, \cdot)\|_{\mathcal{H}^{s+1}} + \|\partial_t v(t, \cdot)\|_{\mathcal{H}^s}$ may be controlled by $C\epsilon$, for another uniform constant $C > 0$, on the interval of existence. The goal would be to obtain existence over an interval of longer length under convenient condition by controlling the Sobolev energy. Our main result is the following:

Theorem 1.2.1. *For any $\rho > 0$, there is a zero measure subset $\mathcal{N}$ of $\mathbb{R}^*_+$ and for every $m \in \mathbb{R}^*_+ - \mathcal{N}$, there are $\epsilon_0 > 0, c > 0, s_0 \in \mathbb{N}$ such that for any $s \geq s_0$, $s \in \mathbb{N}$, $\epsilon \in (0, \epsilon_0)$, any pair (v_0, v_1) of real valued functions belonging to the unit ball of $\mathcal{H}^{s+1}(\mathbb{R}^d) \times \mathcal{H}^s(\mathbb{R}^d)$, problem (1.2.1) has a unique solution*

$$v \in C^0((-T_\epsilon, T_\epsilon), \mathcal{H}^{s+1}(\mathbb{R}^d)) \cap C^1((-T_\epsilon, T_\epsilon), \mathcal{H}^s(\mathbb{R}^d)), \tag{1.2.2}$$

where T_ϵ has a lower bound $T_\epsilon \geq c\epsilon^{-\frac{4}{3}(1-\rho)\kappa}$ if $d \geq 2$ and $T_\epsilon \geq c\epsilon^{-\frac{25}{18}(1-\rho)\kappa}$ if $d = 1$. Moreover, the solution is uniformly bounded in $\mathcal{H}^{s+1}(\mathbb{R}^d)$ on $(-T_\epsilon, T_\epsilon)$ and $\partial_t v$ is uniformly bounded in $\mathcal{H}^s(\mathbb{R}^d)$ on the same interval.

1.3. *A Property of Spectral Projectors on $\mathbb{R}^d$*

As we have pointed out P has eigenvalues given by $\lambda_n = \sqrt{2n + d}$, $n \in \mathbb{N}$. Remark that Π_n is the orthogonal projector of $L^2(\mathbb{R}^d)$ onto the eigenspace associated to λ_n^2. Let us first introduce some notations. For $\xi_0, \xi_1, \ldots, \xi_{p+1}$ $p + 2$ nonnegative real numbers, let $\xi_{i_0}, \xi_{i_1}, \xi_{i_2}$ be respectively the largest, the second largest and the third largest elements among them and ξ' the largest element among $\xi_1, \ldots, \xi_p$, that is,

$$\begin{aligned} &\xi_{i_0} = \max\{\xi_0, \ldots, \xi_{p+1}\}, \quad \xi_{i_1} = \max(\{\xi_0, \ldots, \xi_{p+1}\} - \{\xi_{i_0}\}), \\ &\xi_{i_2} = \max(\{\xi_1, \ldots, \xi_{p+1}\} - \{\xi_{i_0}, \xi_{i_1}\}), \quad \xi' = \max\{\xi_1, \ldots, \xi_p\}. \end{aligned} \tag{1.3.1}$$

Denote

$$\mu(\xi_0, \ldots, \xi_{p+1}) = \left(1 + \sqrt{\xi_{i_1}}\right)\left(1 + \sqrt{\xi_{i_2}}\right). \tag{1.3.2}$$

Set also

$$S(\xi_0, \dots, \xi_{p+1}) = |\xi_{i_0} - \xi_{i_1}| + \mu(\xi_0, \dots, \xi_{p+1}). \tag{1.3.3}$$

The main result of this subsection is the following one:

Theorem 1.3.1. *There is a $\nu \in \mathbb{R}_+^*$, depending only on p ($p \in \mathbb{N}^*$) and dimension d, and for any $N \in \mathbb{N}$, there is a $C_N > 0$ such that for any $n_0, \dots, n_{p+1} \in \mathbb{N}$, any $u_0, \dots, u_{p+1} \in L^2(\mathbb{R}^d)$,*

$$\left| \int \Pi_{n_0} u_0 \cdots \Pi_{n_{p+1}} u_{p+1} dx \right| \le C_N \left(1 + \sqrt{n_{i_2}}\right)^{\nu} \frac{\mu(n_0, \dots, n_{p+1})^N}{S(n_0, \dots, n_{p+1})^N} \prod_{j=0}^{p+1} \|u_j\|_{L^2}. \tag{1.3.4}$$

Furthermore if $d = 1$, we may find for any $\varsigma \in (0, 1)$

$$\left| \int \Pi_{n_0} u_0 \cdots \Pi_{n_{p+1}} u_{p+1} dx \right| \le C_N \frac{(1 + \sqrt{n_{i_2}})^{\nu}}{(1 + \sqrt{n_{i_0}})^{\frac{1}{6}(1-\varsigma)}} \frac{\mu(n_0, \dots, n_{p+1})^N}{S(n_0, \dots, n_{p+1})^N} \prod_{j=0}^{p+1} \|u_j\|_{L^2}. \tag{1.3.5}$$

Proof. By the symmetries we may assume $n_0 \ge n_1 \ge \cdots \ge n_{p+1}$. Then recalling the definition of λ_n in (1.1.1), we only need to show under the condition of Theorem 1.3.1

$$\left| \int \Pi_{n_0} u_0 \cdots \Pi_{n_{p+1}} u_{p+1} dx \right| \le C_N \lambda_{n_2}^{\nu} \frac{(\lambda_{n_1} \lambda_{n_2})^N}{(|\lambda_{n_0}^2 - \lambda_{n_1}^2| + \lambda_{n_1} \lambda_{n_2})^N} \prod_{j=0}^{p+1} \|u_j\|_{L^2} \tag{1.3.6}$$

and when $d = 1$

$$\left| \int \Pi_{n_0} u_0 \cdots \Pi_{n_{p+1}} u_{p+1} dx \right| \le C_N \frac{\lambda_{n_2}^{\nu}}{\lambda_{n_0}^{\frac{1}{6}(1-\varsigma)}} \frac{(\lambda_{n_1} \lambda_{n_2})^N}{(|\lambda_{n_0}^2 - \lambda_{n_1}^2| + \lambda_{n_1} \lambda_{n_2})^N} \prod_{j=0}^{p+1} \|u_j\|_{L^2} \tag{1.3.7}$$

for any $\varsigma \in (0, 1)$. We follow the proof of Proposition 3.6 in [15]. Let A be a linear operator which maps $D(P^{2k})$ into itself. We define a sequence of operators

$$A_N = [P^2, A_{N-1}]; \quad A_0 = A. \tag{1.3.8}$$

Then using integration by parts we have

$$(\lambda_{n_0}^2 - \lambda_{n_1}^2)^N \langle A \Pi_{n_1} u_1, \Pi_{n_0} u_0 \rangle = \langle A_N \Pi_{n_1} u_1, \Pi_{n_0} u_0 \rangle. \tag{1.3.9}$$

Now we set A to be the multiplication operator generated by the function

$$a(x) = (\Pi_{n_2} u_2) \cdots (\Pi_{n_{p+1}} u_{p+1}).$$

Then an induction argument shows

$$A_N = \sum_{|\beta|+|\gamma| \le N,\ |\alpha|+|\beta|+|\gamma| \le 2N} C_{\alpha\beta\gamma} (\partial^\alpha a) x^\beta \partial^\gamma \tag{1.3.10}$$

for constants $C_{\alpha\beta\gamma}$. Therefore we compute for some $v' > \frac{d}{2}$

$$
\begin{aligned}
&|(\lambda_{n_0}^2 - \lambda_{n_1}^2)^N \int (\Pi_{n_0} u_0) \cdots (\Pi_{n_{p+1}} u_{p+1}) dx| \\
&\leq C \sum_{|\beta|+|\gamma|\leq N,\ |\alpha|+|\beta|+|\gamma|\leq 2N} \|(\partial^\alpha a) x^\beta \partial^\gamma \Pi_{n_1} u_1\|_{L^2} \|\Pi_{n_0} u_0\|_{L^2} \\
&\leq C \sum_{|\beta|+|\gamma|\leq N,\ |\alpha|+|\beta|+|\gamma|\leq 2N} \|a\|_{\mathscr{H}^{v'+|\alpha|}} \|\Pi_{n_1} u_1\|_{\mathscr{H}^{|\beta|+|\gamma|}} \|\Pi_{n_0} u_0\|_{L^2},
\end{aligned} \tag{1.3.11}
$$

where in the last estimate we used Definition 1.1.2 and Proposition 1.1.20. Remark that by Definition 1.1.1, one has for any $s \geq 0$

$$
\|\Pi_n u\|_{\mathscr{H}^s} \leq C \lambda_n^s \|\Pi_n u\|_{L^2}. \tag{1.3.12}
$$

This estimate together with the proof of Proposition 1.1.21 gives for $n_2 \geq n_3 \geq \cdots \geq n_{p+1}$

$$
\|a\|_{\mathscr{H}^{v'+|\alpha|}} \leq C \lambda_{n_2}^{v+|\alpha|} \prod_{j=2}^{p+1} \|\Pi_{n_j} u_j\|_{L^2} \tag{1.3.13}
$$

for some $v > 0$ depending only on p and dimension d. Thus we have

$$
\begin{aligned}
&\left|(\lambda_{n_0}^2 - \lambda_{n_1}^2)^N \int (\Pi_{n_0} u_0) \cdots (\Pi_{n_{p+1}} u_{p+1}) dx\right| \\
&\leq C \sum_{|\beta|+|\gamma|\leq N,\ |\alpha|+|\beta|+|\gamma|\leq 2N} \lambda_{n_2}^{v+|\alpha|} \lambda_{n_1}^{|\beta|+|\gamma|} \prod_{j=0}^{p+1} \|\Pi_{n_j} u_j\|_{L^2} \\
&\leq C \sum_{|\alpha|\leq N} \lambda_{n_2}^{v+2N-|\alpha|} \lambda_{n_1}^{|\alpha|} \prod_{j=0}^{p+1} \|\Pi_{n_j} u_j\|_{L^2} \\
&\leq C \lambda_{n_2}^{v+2N} \left(\frac{\lambda_{n_1}}{\lambda_{n_2}}\right)^N \prod_{j=0}^{p+1} \|\Pi_{n_j} u_j\|_{L^2} \\
&\leq C \lambda_{n_2}^{v} (\lambda_{n_1} \lambda_{n_2})^N \prod_{j=0}^{p+1} \|\Pi_{n_j} u_j\|_{L^2}.
\end{aligned} \tag{1.3.14}
$$

Now if $\lambda_{n_1}\lambda_{n_2} \leq |\lambda_{n_0}^2 - \lambda_{n_1}^2|$, then the last estimate implies (1.3.6), while if $\lambda_{n_1}\lambda_{n_2} > |\lambda_{n_0}^2 - \lambda_{n_1}^2|$, then $\frac{\lambda_{n_1}\lambda_{n_2}}{|\lambda_{n_0}^2 - \lambda_{n_1}^2| + \lambda_{n_1}\lambda_{n_2}} \geq \frac{1}{2}$ and thus (1.3.6) is trivially true.

On the other hand, we use the property of the eigenfunctions (see [19]), which in dimension $d = 1$ says that if ϕ_n is the eigenfunction associated to λ_n^2, then one has $\|\phi_n\|_{L^\infty} \leq C \lambda_n^{-\frac{1}{6}}$. Therefore we have

$$
\|\Pi_n u\|_{L^\infty} \leq C \lambda_n^{-\frac{1}{6}} \|\Pi_n u\|_{L^2} \tag{1.3.15}
$$

since in this case the eigenvalues are simple. This estimate gives us

$$\left| \int \Pi_{n_0} u_0 \cdots \Pi_{n_{p+1}} u_{p+1} dx \right| \leq C \lambda_{n_0}^{-\frac{1}{6}} \prod_{j=0}^{p+1} \|\Pi_{n_j} u_j\|_{L^2}. \tag{1.3.16}$$

Combining (1.3.16) with (1.3.6) one gets (1.3.7) for all $N \geq 1$ and some $\nu > 0$ in the case $d = 1$. This concludes the proof. □

2. Long Time Existence

2.1. *Definition and Properties of Multilinear Operators*

Denote by $\mathscr{E}$ the algebraic direct sum of the ranges of the $\Pi_n' s, n \in \mathbb{N}$. With notations (1.3.1)–(1.3.3) we give the following definition.

Definition 2.1.1. Let $\nu \in \mathbb{R}_+, \tau \in \mathbb{R}, p \in \mathbb{N}^*$. We denote by $\mathscr{M}_{p+1}^{\nu,\tau}$ the space of all $p+1$-linear operators $(u_1, \ldots, u_{p+1}) \to M(u_1, \ldots, u_{p+1})$, defined on $\mathscr{E} \times \cdots \times \mathscr{E}$ with values in $L^2(\mathbb{R}^d)$ such that

- For every $(n_0, \ldots, n_{p+1}) \in \mathbb{N}^{p+2}, u_1, \ldots, u_{p+1} \in \mathscr{E}$

$$\Pi_{n_0}[M(\Pi_{n_1} u_1, \ldots, \Pi_{n_{p+1}} u_{p+1})] = 0, \tag{2.1.1}$$

  if $|n_0 - n_{p+1}| > \frac{1}{2}(n_0 + n_{p+1})$ or $n' \overset{def}{=} \max\{n_1, \ldots, n_p\} > n_{p+1}$.
- For any $N \in \mathbb{N}$, there is a $C > 0$ such that for every $(n_0, \ldots, n_{p+1}) \in \mathbb{N}^{p+2}, u_1, \ldots, u_{p+1} \in \mathscr{E}$, one has

$$\begin{aligned} &\|\Pi_{n_0}[M(\Pi_{n_1} u_1, \ldots, \Pi_{n_{p+1}} u_{p+1})]\|_{L^2} \\ &\leq C\left(1 + \sqrt{n_0} + \sqrt{n_{p+1}}\right)^\tau \left(1 + \sqrt{n'}\right)^\nu \frac{\mu(n_0, \ldots, n_{p+1})^N}{S(n_0, \ldots, n_{p+1})^N} \prod_{j=1}^{p+1} \|u_j\|_{L^2}. \end{aligned} \tag{2.1.2}$$

The best constant in the preceding inequality will be denoted by $\|M\|_{\mathscr{M}_{p+1,N}^{\nu,\tau}}$.

We may extend the operators in $\mathscr{M}_{p+1}^{\nu,\tau}$ to Sobolev spaces.

Proposition 2.1.2. *Let $\nu \in \mathbb{R}_+, \tau \in \mathbb{R}, p \in \mathbb{N}^*, s \in \mathbb{N}, s > \nu + 3$. Then any element $M \in \mathscr{M}_{p+1}^{\nu,\tau}$ extends as a bounded operator from $\mathscr{H}^s(\mathbb{R}^d) \times \cdots \times \mathscr{H}^s(\mathbb{R}^d)$ to $\mathscr{H}^{s-\tau-1}(\mathbb{R}^d)$. Moreover, for any $s_0 \in (\nu + 3, s]$, there is $C > 0$ such that for any $M \in \mathscr{M}_{p+1}^{\nu,\tau}$, and any $u_1, \ldots, u_{p+1} \in \mathscr{H}^s(\mathbb{R}^d)$,*

$$\|M(u_1, \ldots, u_{p+1})\|_{\mathscr{H}^{s-\tau-1}} \leq C\|M\|_{\mathscr{M}_{p+1,N}^{\nu,\tau}} \sum_{j=1}^{p+1} \left[\|u_j\|_{\mathscr{H}^s} \prod_{k \neq j} \|u_k\|_{\mathscr{H}^{s_0}} \right]. \tag{2.1.3}$$

Proof. The proof is a modification of Proposition 4.4 in [11]. There is one derivative lost compared to that case. We give it for the convenience of the reader. Using Definition 1.1.1 we write

$$\begin{aligned} &\|M(u_1, \ldots, u_{p+1})\|_{\mathscr{H}^{s-\tau-1}}^2 \\ &\leq C \sum_{n_0} \left\| \sum_{n_1} \cdots \sum_{n_{p+1}} \Pi_{n_0} M(\Pi_{n_1} u_1, \ldots, \Pi_{n_{p+1}} u_{p+1}) \right\|_{L^2}^2 \left(1 + \sqrt{n_0}\right)^{2s-2\tau-2} \end{aligned} \tag{2.1.4}$$

Because of (2.1.1) and using the symmetries we may assume

$$n_0 \sim n_{p+1} \quad \text{and} \quad n_1 \leq \cdots \leq n_p \leq n_{p+1} \leq Cn_0 \tag{2.1.5}$$

when estimating the above quantity. Consequently, we have

$$\begin{aligned} &\mu(n_0, \ldots, n_{p+1}) \sim (1+\sqrt{n_p})(1+\sqrt{n_{p+1}}), \\ &S(n_0, \ldots, n_{p+1}) \sim |n_0 - n_{p+1}| + \mu(n_0, \ldots, n_{p+1}). \end{aligned} \tag{2.1.6}$$

By (2.1.2) the square root of the general term over n_0 sum in (2.1.4) is smaller than

$$C \sum_{n_1 \leq \cdots \leq n_{p+1}} \left(1+\sqrt{n_0}\right)^{s-1} \frac{(1+\sqrt{n_p})^{\nu} \mu(n_0, \ldots, n_{p+1})^N}{S(n_0, \ldots, n_{p+1})^N} \prod_1^{p+1} \|\Pi_{n_j} u_j\|_{L^2}. \tag{2.1.7}$$

We have by (2.1.5) and (2.1.6)

$$\frac{\mu(n_0, \ldots, n_{p+1})}{S(n_0, \ldots, n_{p+1})} \sim \frac{1+\sqrt{n_p}}{|\sqrt{n_0} - \sqrt{n_{p+1}}| + 1 + \sqrt{n_p}}. \tag{2.1.8}$$

The following fact will be useful in this section: For $q \in \mathbb{N}$, $A \geq 1$ and $N > 1$, there is a $C > 0$ independent of q and A such that

$$\sum_{n \in \mathbb{N}} \frac{1}{(|\sqrt{n} - \sqrt{q}| + A)^N} \leq C \frac{1+\sqrt{q}}{A^{N-2}}. \tag{2.1.9}$$

Let $\iota > 2$ be a constant as close to 2 as wanted. Using (2.1.8) and (2.1.9) we deduce

$$\begin{aligned} &\sum_{n_0} \frac{\mu(n_0, \ldots, n_{p+1})^{\iota}}{S(n_0, \ldots, n_{p+1})^{\iota}} \leq C(1+\sqrt{n_{p+1}})(1+\sqrt{n_p})^2, \\ &\sum_{n_{p+1}} \frac{\mu(n_0, \ldots, n_{p+1})^{\iota}}{S(n_0, \ldots, n_{p+1})^{\iota}} \leq C(1+\sqrt{n_0})(1+\sqrt{n_p})^2. \end{aligned} \tag{2.1.10}$$

We estimate the sum over $n_1 \leq \cdots \leq n_{p+1}$ in (2.1.7) by

$$\begin{aligned} &C\Bigg(\sum_{n_1 \leq \cdots \leq n_{p+1}} \frac{(1+\sqrt{n_p})^{\nu} \mu^{\iota}}{S^{\iota}} \prod_{j=1}^{p} \|\Pi_{n_j} u_j\|_{L^2}\Bigg)^{1/2} \\ &\times \Bigg(\sum_{n_1 \leq \cdots \leq n_{p+1}} (1+\sqrt{n_0})^{2s-2}(1+\sqrt{n_p})^{\nu} \frac{\mu^{2N-\iota}}{S^{2N-\iota}} \prod_{j=1}^{p} \|\Pi_{n_j} u_j\|_{L^2} \|\Pi_{n_{p+1}} u_{p+1}\|_{L^2}^2\Bigg)^{1/2}. \end{aligned} \tag{2.1.11}$$

Using (2.1.10) to handle n_{p+1} sum, we bound the first factor in (2.1.11) from above by $C(1+\sqrt{n_0})^{\frac{1}{2}} \Pi_{j=1}^{p} \|u_j\|_{\mathcal{H}^{s_0}}^{\frac{1}{2}}$ if $s_0 > \nu + 3$ using Definition 1.1.1. Incorporating $(1+\sqrt{n_0})^{\frac{1}{2}}$ into the second factor, we have to bound the quantity

$$\Bigg(\sum_{n_1 \leq \cdots \leq n_{p+1}} (1+\sqrt{n_0})^{2s-1}(1+\sqrt{n_p})^{\nu} \frac{\mu^{2N-\iota}}{S^{2N-\iota}} \prod_{j=1}^{p} \|\Pi_{n_j} u_j\|_{L^2} \|\Pi_{n_{p+1}} u_{p+1}\|_{L^2}^2\Bigg)^{1/2}. \tag{2.1.12}$$

By (2.1.5) and $\mu \leq S$ we have

$$\left(1+\sqrt{n_0}\right)^{2s-1}\left(\frac{\mu}{S}\right)^{2N-\iota} \leq C\left(1+\sqrt{n_{p+1}}\right)^{2s-1}\left(\frac{\mu}{S}\right)^{\iota} \tag{2.1.13}$$

if $N > \iota$. Plugging in (2.1.12), (2.1.11) and then (2.1.4) we bound from above the n_0 sum in (2.1.4) by

$$\begin{aligned} C\prod_1^p \|u_j\|_{\mathscr{H}^{s_0}} & \sum_{n_1\leq\cdots\leq n_{p+1}\leq Cn_0} \left(1+\sqrt{n_{p+1}}\right)^{2s-1} \\ & \times \left(1+\sqrt{n_p}\right)^{\nu}\left(\frac{\mu}{S}\right)^{\iota} \prod_{j=1}^{p} \|\Pi_{n_j}u_j\|_{L^2}\|\Pi_{n_{p+1}}u_{p+1}\|_{L^2}^2. \end{aligned} \tag{2.1.14}$$

Changing the order of sums for n_0 and n_{p+1}, we then use (2.1.10) to handle n_0 sum and get a control of (2.1.14) by $C\prod_{j=1}^{p}\|u_j\|_{\mathscr{H}^{s_0}}^2\|u_{p+1}\|_{\mathscr{H}^s}^2$ according to Definition 1.1.1 if $s > \nu + 3$. This concludes the proof. □

Let us define convenient subspaces of the spaces of Definition 2.1.1.

Definition 2.1.3. Let $\nu \in \mathbb{R}_+$, $\tau \in \mathbb{R}$, $p \in \mathbb{N}^*$, $\omega : \{0, \dots, p+1\} \to \{-1, 1\}$ be given.

- If $\sum_{j=0}^{p+1}\omega(j) \neq 0$, we set $\widetilde{\mathscr{M}}_{p+1}^{\nu,\tau}(\omega) = \mathscr{M}_{p+1}^{\nu,\tau}$;
- If $\sum_{j=0}^{p+1}\omega(j) = 0$, we denote by $\widetilde{\mathscr{M}}_{p+1}^{\nu,\tau}(\omega)$ the closed subspace of $\mathscr{M}_{p+1}^{\nu,\tau}$ given by those $M \in \mathscr{M}_{p+1}^{\nu,\tau}$ such that

$$\Pi_{n_0}M(\Pi_{n_1}u_1, \dots, \Pi_{n_{p+1}}u_{p+1}) \equiv 0 \tag{2.1.15}$$

for any $(n_0, \dots, n_{p+1}) \in \mathbb{N}^{p+2}$ such that there is a bijection σ from $\{j; 0 \leq j \leq p+1, \omega(j) = -1\}$ to $\{j; 0 \leq j \leq p+1, \omega(j) = 1\}$ so that for any j in the first set $n_{\sigma(j)} = n_j$.

We shall have to use also classes of remainder operators. If $n_1, \dots, n_{p+1} \in \mathbb{N}$ and $j_0 \in \{1, \dots, p+1\}$ is such that $n_{j_0} = \max\{n_1, \dots, n_{p+1}\}$, we denote

$$\max_2\left(\sqrt{n_1}, \dots, \sqrt{n_{p+1}}\right) = 1 + \max\left\{\sqrt{n_j}; 1 \leq j \leq p+1, j \neq j_0\right\}. \tag{2.1.16}$$

Definition 2.1.4. Let $\nu \in \mathbb{R}_+, \tau \in \mathbb{R}$, $p \in \mathbb{N}^*$. We denote by $\mathscr{R}_{p+1}^{\nu,\tau}$ the space of $\mathbb{C}(p+1)$-linear maps from $\mathscr{E} \times \cdots \times \mathscr{E} \to L^2(\mathbb{R}^d), (u_1, \dots, u_{p+1}) \to R(u_1, \dots, u_{p+1})$ such that for any $N \in \mathbb{N}$, there is a $C > 0$ such that for any $(n_0, \dots, n_{p+1}) \in \mathbb{N}^{p+2}$, any $u_1, \dots, u_{p+1} \in \mathscr{E}$,

$$\begin{aligned} &\|\Pi_{n_0}R(\Pi_{n_1}u_1, \dots, \Pi_{n_{p+1}}u_{p+1})\|_{L^2} \\ &\quad \leq C(1+\sqrt{n_0})^{\tau}\frac{\max_2(\sqrt{n_1}, \dots, \sqrt{n_{p+1}})^{\nu+N}}{(1+\sqrt{n_0}+\cdots+\sqrt{n_{p+1}})^N}\prod_{j=1}^{p+1}\|u_j\|_{L^2}. \end{aligned} \tag{2.1.17}$$

The elements in $\mathscr{R}_{p+1}^{\nu,\tau}$ also extend as bounded operators on Sobolev spaces.

Proposition 2.1.5. *Let $\nu \in \mathbb{R}_+$, $\tau \in \mathbb{R}$, $p \in \mathbb{N}^*$ be given. There is $s_0 \in \mathbb{N}$ such that for any $s \geq s_0$, any $R \in \mathscr{R}^{\nu,\tau}_{p+1}$, $(u_1, \ldots, u_{p+1}) \to R(u_1, \ldots, u_{p+1})$ extends as a bounded map from $\mathscr{H}^s(\mathbb{R}^d) \times \cdots \times \mathscr{H}^s(\mathbb{R}^d) \to \mathscr{H}^{2s-\nu-\tau-7}(\mathbb{R}^d)$. Moreover one has*

$$\|R(u_1, \ldots, u_{p+1})\|_{\mathscr{H}^{2s-\nu-\tau-7}} \leq C \sum_{1\leq j_1<j_2\leq p+1} \left[\|u_{j_1}\|_{\mathscr{H}^s} \|u_{j_2}\|_{\mathscr{H}^s} \prod_{k\neq j_1, k\neq j_2} \|u_k\|_{\mathscr{H}^{s_0}} \right]. \quad (2.1.18)$$

Proof. We may assume $\tau = 0$. By Definition 1.1.1 we have to bound $\|\Pi_{n_0} R(u_1, \ldots, u_{p+1})\|_{L^2}$ from above by $(1 + \sqrt{n_0})^{-2s+\nu+7} c_{n_0}$ for a sequence $(c_{n_0})_{n_0}$ in ℓ^2. To do that we decompose u_j as $\sum_{n_j} \Pi_{n_j} u_j$ and use (2.1.17). By symmetry we limit ourselves to summation over

$$n_1 \leq \cdots \leq n_{p+1}, \quad (2.1.19)$$

from which we deduce

$$\max_2 \left(\sqrt{n_1}, \ldots, \sqrt{n_{p+1}}\right) = 1 + \sqrt{n_p}. \quad (2.1.20)$$

Therefore we are done if we can bound from above

$$C \sum_{n_1\leq\cdots\leq n_{p+1}} \frac{(1+\sqrt{n_p})^{\nu+N}}{(1+\sqrt{n_0}+\cdots+\sqrt{n_{p+1}})^N} \prod_{j=1}^{p-1} (1+\sqrt{n_j})^{-s_0} (1+\sqrt{n_p})^{-s} (1+\sqrt{n_{p+1}})^{-s} \quad (2.1.21)$$

by $(1 + \sqrt{n_0})^{-2s+\nu+7} c_{n_0}$ for s_0, s large enough with $s \geq s_0$ since $\|\Pi_{n_j} u_j\|_{L^2} \leq C(1 + \sqrt{n_j})^{-s} \|u_j\|_{\mathscr{H}^s}$. Using (2.1.19) we get an upper bound of (2.1.21) by

$$C \sum_{n_1\leq\cdots\leq n_{p+1}} \frac{(1+\sqrt{n_p})^{\nu+N-2s}}{(1+\sqrt{n_0}+\sqrt{n_{p+1}})^N} \prod_{j=1}^{p-1} (1+\sqrt{n_j})^{-s_0}. \quad (2.1.22)$$

Using the fact $\sum_{n\in\mathbb{N}} \frac{1}{(\sqrt{n}+A)^N} \leq \frac{C}{A^{N-2}}$ for $N > 2$ and $A \geq 1$, we take the sum over n_{p+1} to get an upper bound of (2.1.21) by

$$C \sum_{n_1\leq\cdots\leq n_p} \frac{(1+\sqrt{n_p})^{\nu+N-2s}}{(1+\sqrt{n_0})^{N-2}} \prod_{j=1}^{p-1} \left(1+\sqrt{n_j}\right)^{-s_0} \quad (2.1.23)$$

if $N > 2$. Now take $N = 2s - \nu - \frac{5}{2}$ and sum over $n_1, \ldots, n_p$. This gives the upper bound we want and thus concludes the proof. □

Definition 2.1.6. Let $\nu \in \mathbb{R}_+$, $\tau \in \mathbb{R}$, $p \in \mathbb{N}^*$, $\omega : \{0, \ldots, p+1\} \to \{-1, 1\}$ be given.

- If $\sum_{j=0}^{p+1} \omega(j) \neq 0$, we set $\widetilde{\mathscr{R}}^{\nu,\tau}_{p+1}(\omega) = \mathscr{R}^{\nu,\tau}_{p+1}$;
- If $\sum_{j=0}^{p+1} \omega(j) = 0$, we denote by $\widetilde{\mathscr{R}}^{\nu,\tau}_{p+1}(\omega)$ the closed subspace of $\mathscr{R}^{\nu,\tau}_{p+1}$ given by those $R \in \mathscr{R}^{\nu,\tau}_{p+1}$ such that

$$\Pi_{n_0} R(\Pi_{n_1} u_1, \ldots, \Pi_{n_{p+1}} u_{p+1}) \equiv 0 \quad (2.1.24)$$

for any $(n_0, \dots, n_{p+1}) \in \mathbb{N}^{p+2}$ such that there is a bijection σ from $\{j; 0 \le j \le p+1, \omega(j) = -1\}$ to $\{j; 0 \le j \le p+1, \omega(j) = 1\}$ so that for any j in the first set $n_{\sigma(j)} = n_j$.

2.2. *Rewriting of the Equation and the Energy*

In this subsection we will write the time derivative of the energy in terms of multilinear operators defined in the previous subsection. To do that, we shall need to analyze the nonlinearity. Decompose

$$-F(v) = -\sum_{p=\kappa}^{2\kappa-1} \frac{\partial_v^{p+1} F(0)}{(p+1)!} v^{p+1} + G(v) \tag{2.2.1}$$

where $G(v)$ vanishes at order $2\kappa+1$ at $v = 0$. (Here we decompose the nonlinearity up to order 2κ for simplicity and it should be enough to decompose it up to order of $[4\kappa/3]+1$). One has

$$cv^{p+1} = c\sum_{n_1} \cdots \sum_{n_{p+1}} (\Pi_{n_1} v) \cdots (\Pi_{n_{p+1}} v)$$

for a real constant c. One may also write this as $A_p(v) \cdot v$ where $A_p(v)$ is an operator of form

$$A_p(v) \cdot w = \sum_{n_1} \cdots \sum_{n_{p+1}} B(n_1, \dots, n_{p+1})(\Pi_{n_1} v) \cdots (\Pi_{n_p} v)(\Pi_{n_{p+1}} w), \tag{2.2.2}$$

where $B(n_1, \dots, n_{p+1})$ is a real valued bounded function supported on $\max\{n_1, \dots, n_p\} \le n_{p+1}$ and B is constant valued on the domain $\max\{n_1, \dots, n_p\} < n_{p+1}$. For instance, when $p = 2$, one may write

$$\begin{aligned}\{(n_1, n_2, n_3); n_j \in \mathbb{N}\} &= \{\max\{n_1, n_2\} \le n_3\} \cup \{n_1 \ge n_2 \text{ and } n_1 > n_3\} \\ &\quad \cup \{n_1 < n_2 \text{ and } n_2 > n_3\}\end{aligned}$$

and

$$\begin{aligned}\sum_{n_1}\sum_{n_2}\sum_{n_3} (\Pi_{n_1} v)(\Pi_{n_2} v)(\Pi_{n_3} v) &= \sum \mathbf{1}_{\{\max\{n_1,n_2\} \le n_3\}} (\Pi_{n_1} v)(\Pi_{n_2} v)(\Pi_{n_3} v) \\ &\quad + \sum \mathbf{1}_{\{n_3 \ge n_2 \text{ and } n_3 > n_1\}} (\Pi_{n_1} v)(\Pi_{n_2} v)(\Pi_{n_3} v) \\ &\quad + \sum \mathbf{1}_{\{n_3 > n_2 \text{ and } n_3 > n_1\}} (\Pi_{n_1} v)(\Pi_{n_2} v)(\Pi_{n_3} v)\end{aligned}$$

using the symmetries, so that in this case

$$B(n_1, n_2, n_3) = c(\mathbf{1}_{\{\max\{n_1,n_2\} \le n_3\}} + \mathbf{1}_{\{n_3 \ge n_2 \text{ and } n_3 > n_1\}} + \mathbf{1}_{\{n_3 > n_2 \text{ and } n_3 > n_1\}}).$$

So if we make a change of unknown $u = (D_t + \Lambda_m)v$ with

$$D_t = -i\partial_t, \quad \Lambda_m = \sqrt{-\Delta + |x|^2 + m^2},$$

we may write using (2.2.1)

$$(D_t - \Lambda_m)u = -\sum_{p=\kappa}^{2\kappa-1} A_p\left(\Lambda_m^{-1}\left(\frac{u+\bar{u}}{2}\right)\right)\Lambda_m^{-1}\left(\frac{u+\bar{u}}{2}\right) + G\left(\Lambda_m^{-1}\left(\frac{u+\bar{u}}{2}\right)\right). \quad (2.2.3)$$

Denote $C(u,\bar{u}) = -\frac{1}{2}\sum_{p=\kappa}^{2\kappa-1} A_p\big(\Lambda_m^{-1}\big(\frac{u+\bar{u}}{2}\big)\big)\Lambda_m^{-1}$ so that

$$(D_t - \Lambda_m)u = C(u,\bar{u})u + C(u,\bar{u})\bar{u} + G\left(\Lambda_m^{-1}\left(\frac{u+\bar{u}}{2}\right)\right). \quad (2.2.4)$$

We have to estimate for the solution u of (2.2.3)

$$\Theta_s(u(t,\cdot)) = \frac{1}{2}\langle \Lambda_m^s u(t,\cdot), \Lambda_m^s u(t,\cdot)\rangle. \quad (2.2.5)$$

Now comes the main result of this subsection:

Proposition 2.2.1. *There are $\nu \in \mathbb{R}_+$ and large enough s_0 such that for any natural number $s \geq s_0$, there are*:

- *Multilinear operators $M_\ell^p \in \widetilde{\mathcal{M}}_{p+1}^{\nu,2s-a}(\omega_\ell)$, $\kappa \leq p \leq 2\kappa-1$, $0 \leq \ell \leq p$ with ω_ℓ defined by $\omega_\ell(j) = -1$, $j = 0,\dots,\ell$, $\omega_\ell(j) = 1$, $j = \ell+1,\dots,p+1$ and $a = 2$ if $d \geq 2$ and $a = \frac{13}{6} - \varsigma$ for any $\varsigma \in (0,1)$ if $d = 1$;*
- *Multilinear operators $\widetilde{M}_\ell^p \in \widetilde{\mathcal{M}}_{p+1}^{\nu,2s-1}(\tilde{\omega}_\ell)$, $\kappa \leq p \leq 2\kappa-1$, $0 \leq \ell \leq p$ with $\tilde{\omega}_\ell$ defined by $\tilde{\omega}_\ell(j) = -1$, $j = 0,\dots,\ell, p+1$, $\tilde{\omega}_\ell(j) = 1$, $j = \ell+1,\dots,p$;*
- *Multilinear operators $R_\ell^p \in \widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\omega_\ell)$, $\widetilde{R}_\ell^p \in \widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\tilde{\omega}_\ell)$, $\kappa \leq p \leq 2\kappa-1$, $0 \leq \ell \leq p$;*
- *A map $u \to T(u)$ defined on $\mathcal{H}^s(\mathbb{R}^d)$ with values in $\mathbb{R}$, satisfying when $\|u\|_{\mathcal{H}^s} \leq 1$, $|T(u)| \leq C\|u\|_{\mathcal{H}^s}^{2\kappa+2}$*

such that

$$\begin{aligned}\frac{d}{dt}\Theta_s(u(t,\cdot)) = &\sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\, i\left\langle M_\ell^p\Big(\underbrace{\bar{u},\dots,\bar{u}}_{\ell}, \underbrace{u,\dots,u}_{p+1-\ell}\Big), u\right\rangle \\ &+ \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\, i\left\langle \widetilde{M}_\ell^p\Big(\underbrace{\bar{u},\dots,\bar{u}}_{\ell}, \underbrace{u,\dots,u}_{p-\ell}, \bar{u}\Big), u\right\rangle \\ &+ \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\, i\left\langle R_\ell^p\Big(\underbrace{\bar{u},\dots,\bar{u}}_{\ell}, \underbrace{u,\dots,u}_{p+1-\ell}\Big), u\right\rangle \\ &+ \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\, i\left\langle \widetilde{R}_\ell^p\Big(\underbrace{\bar{u},\dots,\bar{u}}_{\ell}, \underbrace{u,\dots,u}_{p-\ell}, \bar{u}\Big), u\right\rangle + T(u). \quad (2.2.6)\end{aligned}$$

Proof. We compute according to (2.2.4)

$$\begin{aligned}\frac{d}{dt}\Theta_s(u(t,\cdot)) &= Re\, i\langle \Lambda_m^s D_t u, \Lambda_m^s u\rangle \\ &= Re\, i\langle \Lambda_m^s C(u,\bar{u})u, \Lambda_m^s u\rangle + Re\, i\langle \Lambda_m^s C(u,\bar{u})\bar{u}, \Lambda_m^s u\rangle \\ &\quad + Re\, i\left\langle \Lambda_m^s G\left(\Lambda_m^{-1}\left(\frac{u+\bar{u}}{2}\right)\right), \Lambda_m^s u\right\rangle. \quad (2.2.7)\end{aligned}$$

The last term in the right hand side of (2.2.7) contributes to the last term in (2.2.6) by Proposition 1.1.21. Let us treat the other two terms in the right hand side of (2.2.7). □

Lemma 2.2.2. *There are $M_\ell^p \in \widetilde{\mathcal{M}}_{p+1}^{\nu,2s-a}(\omega_\ell)$, $R_\ell^p \in \widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\omega_\ell)$, $\kappa \le p \le 2\kappa - 1$, $0 \le \ell \le p$ with ω_ℓ defined by $\omega_\ell(j) = -1$, $j = 0, \dots, \ell$, $\omega_\ell(j) = 1$, $j = \ell + 1, \dots, p + 1$ and $a = 2$ if $d \ge 2$ and $a = \frac{13}{6} - \varsigma$ for any $\varsigma \in (0, 1)$ if $d = 1$, such that*

$$Re\, i\langle \Lambda_m^s C(u, \bar{u})u, \Lambda_m^s u\rangle = \sum_{p=\kappa}^{2\kappa-1} \sum_{\ell=0}^{p} Re\, i\Big\langle M_\ell^p\Big(\underbrace{\bar{u}, \dots, \bar{u}}_{\ell}, \underbrace{u, \dots, u}_{p+1-\ell}\Big), u\Big\rangle$$
$$+ \sum_{p=\kappa}^{2\kappa-1} \sum_{\ell=0}^{p} Re\, i\Big\langle R_\ell^p\Big(\underbrace{\bar{u}, \dots, \bar{u}}_{\ell}, \underbrace{u, \dots, u}_{p+1-\ell}\Big), u\Big\rangle. \tag{2.2.8}$$

Proof of Lemma 2.2.2. Let χ be a cut-off function near 0 with small support and λ_n defined in (1.1.1). We may decompose the operator $A_p(v)$ defined in (2.2.2) as

$$A_p(v) = A_p^1(v) + A_p^2(v) + A_p^3(v), \tag{2.2.9}$$

where $A_p^j(v)(j = 1, 2, 3)$ are operators of form

$$A_p^1(v) \cdot w = \sum_{n_0} \cdots \sum_{n_{p+1}} B_1(n_0, \dots, n_{p+1})\Pi_{n_0}[(\Pi_{n_1} v) \cdots (\Pi_{n_p} v)(\Pi_{n_{p+1}} w)], ,$$
$$A_p^2(v) \cdot w = \sum_{n_0} \cdots \sum_{n_{p+1}} B_2(n_0, \dots, n_{p+1})\Pi_{n_0}[(\Pi_{n_1} v) \cdots (\Pi_{n_p} v)(\Pi_{n_{p+1}} w)],$$
$$A_p^3(v) \cdot w = \sum_{n_1} \cdots \sum_{n_{p+1}} B_3(n_1, \dots, n_{p+1})\Pi_{n_0}[(\Pi_{n_1} v) \cdots (\Pi_{n_p} v)(\Pi_{n_{p+1}} w)], \tag{2.2.10}$$

with

$$B_1(n_0, \dots, n_{p+1}) = B(n_1, \dots, n_{p+1})\chi\left(\frac{|\lambda_{n_0}^2 - \lambda_{n_{p+1}}^2|}{\lambda_{n_0}^2 + \lambda_{n_{p+1}}^2}\right)\mathbf{1}_{\{\max\{n_1,\dots,n_p\}<\delta n_{p+1}\}},$$
$$B_2(n_0, \dots, n_{p+1}) = B(n_1, \dots, n_{p+1})\left(1 - \chi\left(\frac{|\lambda_{n_0}^2 - \lambda_{n_{p+1}}^2|}{\lambda_{n_0}^2 + \lambda_{n_{p+1}}^2}\right)\right)\mathbf{1}_{\{\max\{n_1,\dots,n_p\}<\delta n_{p+1}\}}, \tag{2.2.11}$$
$$B_3(n_1, \dots, n_{p+1}) = B(n_1, \dots, n_{p+1})\mathbf{1}_{\{\max\{n_1,\dots,n_p\}\ge\delta n_{p+1}\}},$$

with some small $\delta > 0$. Therefore for the operator $C(u, \bar{u})$ defined above (2.2.4), we have

$$C(u, \bar{u}) = -\frac{1}{2}\sum_{j=1}^{3}\sum_{p=\kappa}^{2\kappa-1} A_p^j\left(\Lambda_m^{-1}\left(\frac{u + \bar{u}}{2}\right)\right)\Lambda_m^{-1}. \tag{2.2.12}$$

So the left hand side of (2.2.8) may be written as

$$-\frac{1}{2}\sum_{j=1}^{3}\sum_{p=\kappa}^{2\kappa-1} Re\, i\left\langle \Lambda_m^{2s} A_p^j\left(\Lambda_m^{-1}\left(\frac{u + \bar{u}}{2}\right)\right)\Lambda_m^{-1} u, u\right\rangle := \sum_{j=1}^{3}\sum_{p=\kappa}^{2\kappa-1} I_p^j. \tag{2.2.13}$$

Let us treat these quantities term by term.

(i) The term I_p^1.
Note that $-4I_p^1$ equals to

$$Re\, i\left\langle \left[\Lambda_m^{2s} A_p^1\left(\Lambda_m^{-1}\left(\frac{u+\bar{u}}{2}\right)\right)\Lambda_m^{-1} - \left(A_p^1\left(\Lambda_m^{-1}\left(\frac{u+\bar{u}}{2}\right)\right)\Lambda_m^{-1}\right)^* \Lambda_m^{2s}\right]u, u\right\rangle, \quad (2.2.14)$$

which may be written as

$$\begin{aligned} &Re\, i\left\langle \left[\Lambda_m^{2s}, A_p^1\left(\Lambda_m^{-1}\left(\frac{u+\bar{u}}{2}\right)\right)\Lambda_m^{-1}\right]u, u\right\rangle \\ &\quad + Re\, i\left\langle \left[A_p^1\left(\Lambda_m^{-1}\left(\frac{u+\bar{u}}{2}\right)\right)\Lambda_m^{-1} - \left(A_p^1\left(\Lambda_m^{-1}\left(\frac{u+\bar{u}}{2}\right)\right)\Lambda_m^{-1}\right)^*\right]\Lambda_m^{2s}u, u\right\rangle \\ &:= I + II \end{aligned} \quad (2.2.15)$$

We expand the first term in (2.2.15) using (2.2.10) to get

$$\begin{aligned} I &= Re\, i\left\langle \sum_{n\in\mathbb{N}^{p+2}} \pi_1 \Pi_{n_0}\left[\left(\Pi_{n_1}\Lambda_m^{-1}\left(\frac{u+\bar{u}}{2}\right)\right)\cdots\left(\Pi_{n_p}\Lambda_m^{-1}\left(\frac{u+\bar{u}}{2}\right)\right)\left(\Pi_{n_{p+1}}\Lambda_m^{-1}u\right)\right], u\right\rangle \\ &= Re\, i\left\langle \sum_{n\in\mathbb{N}^{p+2}} \sum_{\ell=0}^{p} \pi_2 \Pi_{n_0}[(\Pi_{n_1}\Lambda_m^{-1}\bar{u})\cdots(\Pi_{n_\ell}\Lambda_m^{-1}\bar{u})(\Pi_{n_{\ell+1}}\Lambda_m^{-1}u)\cdots(\Pi_{n_{p+1}}\Lambda_m^{-1}u)], u\right\rangle \\ &= Re\, i \sum_{n\in\mathbb{N}^{p+2}} \sum_{\ell=0}^{p} \pi_2 \int (\Pi_{n_0}\bar{u})(\Pi_{n_1}\Lambda_m^{-1}\bar{u})\cdots(\Pi_{n_\ell}\Lambda_m^{-1}\bar{u})(\Pi_{n_{\ell+1}}\Lambda_m^{-1}u)\cdots(\Pi_{n_{p+1}}\Lambda_m^{-1}u)dx, \end{aligned} \quad (2.2.16)$$

where we have used notations

$$\begin{aligned} n &= (n_0, \ldots, n_{p+1}), \\ \pi_1 &= B_1(n_0, \ldots, n_{p+1})[(m^2+\lambda_{n_0}^2)^s - (m^2+\lambda_{n_{p+1}}^2)^s], \\ \pi_2 &= \frac{1}{2^p}\binom{p}{\ell} B_1(n_0, \ldots, n_{p+1})[(m^2+\lambda_{n_0}^2)^s - (m^2+\lambda_{n_{p+1}}^2)^s]. \end{aligned} \quad (2.2.17)$$

Let ω_ℓ be defined in the statement of the lemma and set

$$\begin{aligned} S_p^\ell = \big\{&(n_0, \ldots, n_{p+1}) \in \mathbb{N}^{p+2};\ \text{there exists a bijection } \sigma \text{ from} \\ &\{j; 0 \le j \le p+1, \omega_\ell(j) = -1\} \text{ to } \{j; 0 \le j \le p+1, \omega_\ell(j) = 1\} \\ &\text{such that for each } j \text{ in the first set } n_j = n_{\sigma(j)}\big\}. \end{aligned} \quad (2.2.18)$$

Now we look at the integral in the last line of (2.2.16). If $n \in S_p^\ell$ with $S_p^\ell \neq \emptyset$, there is a bijection σ from $\{0, \ldots, \ell\}$ to $\{\ell, \ldots, p+1\}$ such that $n_j = n_{\sigma(j)}$, $j = 0, \ldots, \ell$. So we may couple $\Pi_{n_j}\bar{u}$, $j = 0, \ldots, \ell$ with $\Pi_{n_{\sigma(j)}}u$, $j = 0, \ldots, \ell$. Since π_2 is real, we get zero if we take the sum over $n \in S_p^\ell$ when computing the right hand side of (2.2.16). Therefore we may assume $n \notin S_p^\ell$ when computing I. Now we define

$$M_\ell^{p,1}(u_1, \dots, u_{p+1}) = -\frac{1}{4} \sum_{n \notin S_p^\ell} \pi_2 \Pi_{n_0}[(\Pi_{n_1} \Lambda_m^{-1} u_1) \cdots (\Pi_{n_{p+1}} \Lambda_m^{-1} u_{p+1})]. \tag{2.2.19}$$

It follows from the second equality in (2.2.16) that

$$I = -4 \sum_{\ell=0}^{p} Re\, i \left\langle M_\ell^{p,1}\Big(\underbrace{\bar{u}, \dots, \bar{u}}_{\ell}, \underbrace{u, \dots, u}_{p+1-\ell}\Big), u \right\rangle. \tag{2.2.20}$$

Let us turn to the term II in (2.2.15). Note that $A_p^1(v)^*$ is an operator of form

$$A_p^1(v)^* \cdot w = \sum_{n \in \mathbb{N}^{p+2}} B_1(n_{p+1}, n_1, \dots, n_p, n_0) \Pi_{n_0}[(\Pi_{n_1} v) \cdots (\Pi_{n_p} v)(\Pi_{n_{p+1}} w)]. \tag{2.2.21}$$

Thus we may compute using (2.2.10)

$$\begin{aligned} II &= Re\, i \Big\langle \sum_{n \in \mathbb{N}^{p+2}} \sum_{\ell=0}^{p} \pi_3 \Pi_{n_0}[(\Pi_{n_1} \Lambda_m^{-1} \bar{u}) \cdots (\Pi_{n_\ell} \Lambda_m^{-1} \bar{u})(\Pi_{n_{\ell+1}} \Lambda_m^{-1} u) \cdots \\ &\qquad \times (\Pi_{n_p} \Lambda_m^{-1} u)(\Pi_{n_{p+1}} \Lambda_m^{2s} u)], u \Big\rangle \\ &= Re\, i \sum_{n \in \mathbb{N}^{p+2}} \sum_{\ell=0}^{p} \pi_3 \int (\Pi_{n_0} \bar{u})(\Pi_{n_1} \Lambda_m^{-1} \bar{u}) \cdots (\Pi_{n_\ell} \Lambda_m^{-1} \bar{u})(\Pi_{n_{\ell+1}} \Lambda_m^{-1} u) \cdots \\ &\quad \times (\Pi_{n_p} \Lambda_m^{-1} u)(\Pi_{n_{p+1}} \Lambda_m^{2s} u) dx, \end{aligned} \tag{2.2.22}$$

where

$$\begin{aligned} \pi_3 = \frac{1}{2^p} \binom{p}{\ell} \big[& B_1(n_0, n_1, \dots, n_p, n_{p+1})(m^2 + \lambda_{n_{p+1}}^2)^{-\frac{1}{2}} \\ & - B_1(n_{p+1}, n_1, \dots, n_p, n_0)(m^2 + \lambda_{n_0}^2)^{-\frac{1}{2}} \big]. \end{aligned} \tag{2.2.23}$$

With the same reasoning as in the paragraph above (2.2.19) we get zero if we take the sum over $n \in S_p^\ell$ when computing the right hand side of (2.2.22). So we may assume $n \notin S_p^\ell$ and define

$$M_\ell^{p,2}(u_1, \dots, u_{p+1}) = -\frac{1}{4} \sum_{n \notin S_p^\ell} \pi_3 \Pi_{n_0}\big[(\Pi_{n_1} \Lambda_m^{-1} u_1) \cdots (\Pi_{n_p} \Lambda_m^{-1} u_p)(\Pi_{n_{p+1}} \Lambda_m^{2s} u_{p+1})\big]. \tag{2.2.24}$$

It follows from (2.2.22) that

$$II = -4 \sum_{\ell=0}^{p} Re\, i \left\langle M_\ell^{p,2}(\underbrace{\bar{u}, \dots, \bar{u}}_{\ell}, \underbrace{u, \dots, u}_{p+1-\ell}), u \right\rangle. \tag{2.2.25}$$

Let us check that $M_\ell^{p,1}, M_\ell^{p,2} \in \widetilde{\mathscr{M}}_{p+1}^{\nu,2s-a}(\omega_\ell)$ for some $\nu > 0$, where $a = 2$ if $d \geq 2$ and $a = \frac{13}{6} - \varsigma$ for any $\varsigma \in (0, 1)$ if $d = 1$. Since the function $B_1(n_0, \ldots, n_{n_{p+1}})$ is supported on domain $n' = \max\{n_1, \ldots, n_p\} < \delta n_{p+1}$ and $n_0 \sim n_{p+1}$ (this is because of the cut-off function and (1.1.1)), we see that (2.1.1) holds true if *supp* χ and δ are small. Let us use Theorem 1.3.1 to show that (2.1.2) holds true with $\tau = 2s - a$ for $M_\ell^{p,1}$ and $M_\ell^{p,2}$. Remark that we have

$$|\pi_2| \leq C\big(1 + |\sqrt{n_0} - \sqrt{n_{p+1}}|\big)\big(1 + \sqrt{n_0} + \sqrt{n_{p+1}}\big)^{2s-1}, \tag{2.2.26}$$

$$|\pi_3| \leq C\big(1 + \sqrt{n'}\big)^2\big(1 + |\sqrt{n_0} - \sqrt{n_{p+1}}|\big)\big(1 + \sqrt{n_0} + \sqrt{n_{p+1}}\big)^{-2}. \tag{2.2.27}$$

Indeed, (2.2.26) follows from the fact

$$|(m^2 + \lambda_{n_0}^2)^s - (m^2 + \lambda_{n_{p+1}}^2)^s| \leq C(|\lambda_{n_0} - \lambda_{n_{p+1}}|)(1 + \lambda_{n_0} + \lambda_{n_{p+1}})^{2s-1}.$$

If $n' < \delta n_0$ and $n' < \delta n_{p+1}$ for small $\delta > 0$, then

$$B_1(n_0, n_1, \ldots, n_p, n_{p+1}) = B_1(n_{p+1}, n_1, \ldots, n_p, n_0)$$

since $B(n_1, \ldots, n_{p+1})$ is constant valued on the domain $n' < n_{p+1}$. Thus (2.2.27) follows from the fact

$$\big|(m^2 + \lambda_{n_0}^2)^{-\frac{1}{2}} - (m^2 + \lambda_{n_{p+1}}^2)^{-\frac{1}{2}}\big| \leq C(|\lambda_{n_0} - \lambda_{n_{p+1}}|)(1 + \lambda_{n_0} + \lambda_{n_{p+1}})^{-2}.$$

Otherwise, assume $n' \geq \delta n_0$ or $n' \geq \delta n_{p+1}$. Then we must have $n' \geq Cn_0$ and $n' \geq Cn_{p+1}$ if B_1 is non zero, since $n_0 \sim n_{p+1}$ which is because of the cut-off function. In this case, (2.2.27) holds true trivially.

Moreover, on the support of $\Pi_{n_0} M_\ell^{p,l}(\Pi_{n_1} u_1, \ldots, \Pi_{n_{p+1}} u_{p+1})(l = 1, 2)$, i.e., $n_0 \sim n_{p+1}$ and $n_{p+1} \geq \max\{n_1, \ldots, n_p\} = n'$, we have

$$\begin{aligned} 1 + \sqrt{n_{i_2}} \sim 1 + \sqrt{n'}, \quad \mu(n_0, \ldots, n_{p+1}) &\sim \big(1 + \sqrt{n_{p+1}}\big)\big(1 + \sqrt{n'}\big), \\ S(n_0, \ldots, n_{p+1}) &\sim |n_0 - n_{p+1}| + \big(1 + \sqrt{n_{p+1}}\big)\big(1 + \sqrt{n'}\big), \end{aligned} \tag{2.2.28}$$

from which we deduce

$$\frac{\mu(n_0, \ldots, n_{p+1})}{S(n_0, \ldots, n_{p+1})} \sim \frac{1 + \sqrt{n'}}{|\sqrt{n_0} - \sqrt{n_{p+1}}| + 1 + \sqrt{n'}}. \tag{2.2.29}$$

Thus

$$\big(1 + |\sqrt{n_0} - \sqrt{n_{p+1}}|\big)\frac{\mu(n_0, \ldots, n_{p+1})}{S(n_0, \ldots, n_{p+1})} \leq C\big(1 + \sqrt{n'}\big).$$

Then we use Theorem 1.3.1 (with dimension $d \geq 2$) to get for $l = 1, 2$

$$\begin{aligned} &\|\Pi_{n_0} M_\ell^{p,l}(\Pi_{n_1} u_1, \ldots, \Pi_{n_{p+1}} u_{p+1})\|_{L^2} \\ &\quad \leq C\big(1 + \sqrt{n_0} + \sqrt{n_{p+1}}\big)^{2s-2}\big(1 + \sqrt{n'}\big)^{\nu+2}\big(1 + |\sqrt{n_0} - \sqrt{n_{p+1}}|\big) \end{aligned}$$

$$\times \frac{\mu(n_0,\ldots,n_{p+1})^N}{S(n_0,\ldots,n_{p+1})^N}\prod_{j=1}^{p+1}\|u_j\|_{L^2}$$

$$\leq C\big(1+\sqrt{n_0}+\sqrt{n_{p+1}}\big)^{2s-2}(1+\sqrt{n'})^{\nu+3}\frac{\mu(n_0,\ldots,n_{p+1})^{N-1}}{S(n_0,\ldots,n_{p+1})^{N-1}}\prod_{j=1}^{p+1}\|u_j\|_{L^2}. \tag{2.2.30}$$

So $M_\ell^{p,l}\in\mathscr{M}_\ell^{\nu,2s-2}$ for some other $\nu>0$ in dimension $d\geq 2$. The case of dimension one is similar (2.1.15) with $\omega=\omega_\ell$ is satisfied by definition. Thus $M_\ell^{p,1},M_\ell^{p,2}\in\widetilde{\mathscr{M}}_{p+1}^{\nu,2s-a}(\omega_\ell)$ and we have proved

$$I_p^1=\sum_{\ell=0}^{p}Re\,i\Big\langle M_\ell^{p,1}\Big(\underbrace{\bar u,\ldots,\bar u}_{\ell},\underbrace{u,\ldots,u}_{p+1-\ell}\Big),u\Big\rangle$$
$$+\sum_{\ell=0}^{p}Re\,i\Big\langle M_\ell^{p,2}\Big(\underbrace{\bar u,\ldots,\bar u}_{\ell},\underbrace{u,\ldots,u}_{p+1-\ell}\Big),u\Big\rangle. \tag{2.2.31}$$

(ii) The term I_p^2.
Using (2.2.10) we get

$$-2I_p^2=Re\,i\Big\langle\sum_{n\in\mathbb{N}^{p+2}}\sum_{\ell=0}^{p}\pi_4\Lambda_m^{2s}\Pi_{n_0}[(\Pi_{n_1}\Lambda_m^{-1}\bar u)\cdots(\Pi_{n_\ell}\Lambda_m^{-1}\bar u)(\Pi_{n_{\ell+1}}\Lambda_m^{-1}u)\cdots$$
$$\times(\Pi_{n_{p+1}}\Lambda_m^{-1}u)],u\Big\rangle$$
$$=Re\,i\sum_{n\in\mathbb{N}^{p+2}}\sum_{\ell=0}^{p}\pi_4\int(\Pi_{n_0}\Lambda_m^{2s}\bar u)(\Pi_{n_1}\Lambda_m^{-1}\bar u)\cdots(\Pi_{n_\ell}\Lambda_m^{-1}\bar u)(\Pi_{n_{\ell+1}}\Lambda_m^{-1}u)\cdots$$
$$\times(\Pi_{n_{p+1}}\Lambda_m^{-1}u)dx \tag{2.2.32}$$

where

$$\pi_4=\frac{1}{2^p}\binom{p}{\ell}B_2(n_0,\ldots,n_{p+1}).$$

We may rule out the sum over $n\in S_p^\ell$ in the above computation with the same reasoning as in the paragraph above (2.2.19). Thus if we define

$$R_\ell^{p,1}(u_1,\ldots,u_{p+1})=-\frac{1}{2}\sum_{n\notin S_p^\ell}\pi_4\Lambda_m^{2s}\Pi_{n_0}[(\Pi_{n_1}\Lambda_m^{-1}u_1)\cdots(\Pi_{n_{p+1}}\Lambda_m^{-1}u_{p+1})], \tag{2.2.33}$$

we have

$$I_p^2=\sum_{\ell=0}^{p}Re\,i\Big\langle R_\ell^{p,1}\Big(\underbrace{\bar u,\ldots,\bar u}_{\ell},\underbrace{u,\ldots,u}_{p+1-\ell}\Big),u\Big\rangle. \tag{2.2.34}$$

From the support property of function $B_2(n_0,\ldots,n_{p+1})$ we know that $\Pi_{n_0}R_\ell^{p,1}(\Pi_{n_1}u_1,\ldots,\Pi_{n_{p+1}}u_{p+1})$ is supported on $\max\{n_1,\ldots,n_p\}<\delta n_{p+1}$ and

$|n_0 - n_{p+1}| \geq c(n_0 + n_{p+1})$ for some small $c > 0$. Therefore, on its support, if $n_0 > Cn_{p+1}$ for a large C, we have

$$\mu(n_0, \ldots, n_{p+1}) = \left(1 + \sqrt{n_{p+1}}\right)\left(1 + \sqrt{n'}\right) \leq \left(1 + \sqrt{n_0}\right)\left(1 + \sqrt{n'}\right),$$
$$S(n_0, \ldots, n_{p+1}) = |n_0 - n_{p+1}| + \left(1 + \sqrt{n_{p+1}}\right)\left(1 + \sqrt{n'}\right) \sim \left(1 + \sqrt{n_0}\right)^2$$

and if $n_0 \leq Cn_{p+1}$, we have

$$\mu(n_0, \ldots, n_{p+1}) \leq (1 + \sqrt{n'})(1 + \sqrt{n_{p+1}}),$$
$$S(n_0, \ldots, n_{p+1}) \geq c(|n_0 - n_{p+1}|) \geq c(n_0 + n_{p+1}) \sim (1 + \sqrt{n_{p+1}})^2.$$

In both cases we have

$$\frac{\mu(n_0, \ldots, n_{p+1})}{S(n_0, \ldots, n_{p+1})} \leq C\frac{1 + \sqrt{n'}}{1 + \sqrt{n_0} + \cdots + \sqrt{n_{p+1}}} = C\frac{\max_2(\sqrt{n_1}, \ldots, \sqrt{n_{p+1}})}{1 + \sqrt{n_0} + \cdots + \sqrt{n_{p+1}}}, \quad (2.2.35)$$

where $\max_2(\sqrt{n_1}, \ldots, \sqrt{n_{p+1}})$ is defined above Definition 2.1.4. Thus Theorem 1.3.1 allows us to get (2.1.17) with $\tau = 2s$ and some $v > 0$. (2.1.24) with $\omega = \omega_\ell$ is satisfied by the definition of $R_\ell^{p,1}$. So $R_\ell^{p,1} \in \widetilde{\mathcal{R}}_{p+1}^{v,2s}(\omega_\ell)$.

(iii) The term I_p^3.

The treatment of I_p^3 is similar to that of I_p^2. The only difference is that we have different support for B_2 and B_3. So we define

$$R_\ell^{p,2}(u_1, \ldots, u_{p+1}) = -\frac{1}{2}\sum_{n \notin S_p^\ell} \pi_5 \Lambda_m^{2s} \Pi_{n_0}[(\Pi_{n_1}\Lambda_m^{-1}u_1)\cdots(\Pi_{n_{p+1}}\Lambda_m^{-1}u_{p+1})] \quad (2.2.36)$$

with π_5 given by

$$\pi_5 = \frac{1}{2^p}\binom{p}{\ell} B_3(n_1, \ldots, n_{p+1}) \quad (2.2.37)$$

and we get

$$I_p^3 = \sum_{\ell=0}^{p} Re\, i\left\langle R_\ell^{p,2}\left(\underbrace{\bar{u}, \ldots, \bar{u}}_{\ell}, \underbrace{u, \ldots, u}_{p+1-\ell}\right), u\right\rangle. \quad (2.2.38)$$

From the support property of B_3 we know that $\Pi_{n_0} R_\ell^{p,2}(\Pi_{n_1}u_1, \ldots, \Pi_{n_{p+1}}u_{p+1})$ is supported on domain $\delta n_{p+1} \leq \max\{n_1, \ldots, n_p\} = n' \leq n_{p+1}$. So on this domain we have

$$\mu(n_0, \ldots, n_{p+1}) \leq \left(1 + \sqrt{n_{p+1}}\right)\left(1 + \sqrt{n'}\right),$$
$$S(n_0, \ldots, n_{p+1}) \sim \left(1 + \sqrt{n_0} + \sqrt{n_{p+1}}\right)^2,$$

from which we deduce

$$\frac{\mu(n_0, \ldots, n_{p+1})}{S(n_0, \ldots, n_{p+1})} \leq C\frac{1 + \sqrt{n'}}{1 + \sqrt{n_0} + \cdots + \sqrt{n_{p+1}}}. \quad (2.2.39)$$

Thus we have by Theorem 1.3.1, for any $N \in \mathbb{N}$, there exists $C_N > 0$, such that (2.1.17) holds true with $\tau = 2s$ and some $\nu > 0$. On the other hand, (2.1.24) with $\omega = \omega_\ell$ is satisfied by the definition. So $R_\ell^{p,2} \in \widetilde{\mathscr{R}}_{p+1}^{\nu,2s}(\omega_\ell)$.

Taking M_ℓ^p to be the sum of $M_\ell^{p,1}$ and $M_\ell^{p,2}$, and R_ℓ^p the sum of $R_\ell^{p,1}$ and $R_\ell^{p,2}$, we get (2.2.8) with $M_\ell^p \in \widetilde{\mathscr{M}}_{p+1}^{\nu,2s-a}(\omega_\ell)$ and $R_\ell^p \in \widetilde{\mathscr{R}}_{p+1}^{\nu,2s}(\omega_\ell)$. This concludes the proof of the lemma. $\square$

We have to treat the second term in the right hand side of (2.2.7).

Lemma 2.2.3. *There are multilinear operators* $\widetilde{M}_\ell^p \in \widetilde{\mathscr{M}}_{p+1}^{\nu,2s-1}(\tilde{\omega}_\ell)$, $\widetilde{R}_\ell^p \in \widetilde{\mathscr{R}}_{p+1}^{\nu,2s}(\tilde{\omega}_\ell)$, $\kappa \le p \le 2\kappa - 1$, $0 \le \ell \le p$ *with* $\tilde{\omega}_\ell$ *defined by* $\tilde{\omega}_\ell(j) = -1$, $j = 0, \ldots, \ell, p+1$, $\tilde{\omega}_\ell(j) = 1$, $j = \ell+1, \ldots, p$, *such that*

$$Re\, i\langle \Lambda_m^s C(u,\bar{u})\bar{u}, \Lambda_m^s u\rangle = \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\, i\Big\langle \widetilde{M}_\ell^p\Big(\underbrace{\bar{u},\ldots,\bar{u}}_{\ell}, \underbrace{u,\ldots,u}_{p-\ell}, \bar{u}\Big), u\Big\rangle$$
$$+ \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\, i\Big\langle \widetilde{R}_\ell^p\Big(\underbrace{\bar{u},\ldots,\bar{u}}_{\ell}, \underbrace{u,\ldots,u}_{p-\ell}, \bar{u}\Big), u\Big\rangle. \quad (2.2.40)$$

Proof of Lemma 2.2.3. Let $\tilde{\omega}_\ell$ be defined in the statement of the lemma. We set

$$\begin{aligned}\widetilde{S}_p^\ell = \big\{&(n_0,\ldots,n_{p+1}) \in \mathbb{N}^{p+2};\ \text{there exists bijection } \sigma \text{ from}\\ &\{j; 0 \le j \le p+1, \tilde{\omega}_\ell(j) = -1\} \text{ to } \{j; 0 \le j \le p+1, \tilde{\omega}_\ell(j) = 1\}\\ &\text{such that for each } j \text{ in the first set } n_j = n_{\sigma(j)}\big\}. \end{aligned} \quad (2.2.41)$$

Taking the expression of $C(u,\bar{u})$ defined above (2.2.4) into account, we compute using notation (2.2.2)

$$\begin{aligned} Re\, i\langle \Lambda_m^{2s} C(u,\bar{u})\bar{u}, u\rangle &= Re\, i\Big\langle -\frac{1}{2}\sum_{p=\kappa}^{2\kappa-1} \Lambda_m^{2s} A_p\Big(\Lambda_m^{-1}\Big(\frac{u+\bar{u}}{2}\Big)\Big)\Lambda_m^{-1}\bar{u}, u\Big\rangle\\ &= Re\, i\Big\langle \sum_{p=\kappa}^{2\kappa-1}\sum_{n\in\mathbb{N}^{p+2}}\sum_{\ell=0}^{p} \pi_6 \Lambda_m^{2s}\Pi_{n_0}[(\Pi_{n_1}\Lambda_m^{-1}\bar{u})\cdots(\Pi_{n_\ell}\Lambda_m^{-1}\bar{u})\\ &\qquad \times (\Pi_{n_{\ell+1}}\Lambda_m^{-1}u)\cdots(\Pi_{n_p}\Lambda_m^{-1}u)(\Pi_{n_{p+1}}\Lambda_m^{-1}\bar{u})], u\Big\rangle\\ &= Re\, i\sum_{p=\kappa}^{2\kappa-1}\sum_{n\in\mathbb{N}^{p+2}}\sum_{\ell=0}^{p}\pi_6\int(\Pi_{n_0}\Lambda_m^{2s}\bar{u})(\Pi_{n_1}\Lambda_m^{-1}\bar{u})\cdots(\Pi_{n_\ell}\Lambda_m^{-1}\bar{u})\\ &\qquad \times (\Pi_{n_{\ell+1}}\Lambda_m^{-1}u)\cdots(\Pi_{n_p}\Lambda_m^{-1}u)(\Pi_{n_{p+1}}\Lambda_m^{-1}\bar{u})\,dx, \end{aligned} \quad (2.2.42)$$

where π_6 is given by

$$\pi_6 = -\frac{1}{2^{p+1}}\binom{p}{\ell} B(n_1,\ldots,n_{p+1}). \quad (2.2.43)$$

With the same reasoning as in the paragraph above (2.2.19) we may assume $n \notin \widetilde{S}_p^\ell$ in the computation of (2.2.42). Let $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ near zero, and $\operatorname{supp}\chi$ small enough. According to (2.2.42), we define

$$\widetilde{M}_\ell^p(u_1, \dots, u_{p+1}) = \sum_{n \notin \widetilde{S}_p^\ell} \chi\left(\frac{|\lambda_{n_0}^2 - \lambda_{n_{p+1}}^2|}{\lambda_{n_0}^2 + \lambda_{n_{p+1}}^2}\right)\pi_6 \Lambda_m^{2s}\Pi_{n_0}[(\Pi_{n_1}\Lambda_m^{-1}u_1)\cdots(\Pi_{n_{p+1}}\Lambda_m^{-1}u_{p+1})],$$

$$\widetilde{R}_\ell^p(u_1, \dots, u_{p+1}) = \sum_{n \notin \widetilde{S}_p^\ell}\left(1 - \chi\left(\frac{|\lambda_{n_0}^2 - \lambda_{n_{p+1}}^2|}{\lambda_{n_0}^2 + \lambda_{n_{p+1}}^2}\right)\right)\pi_6 \Lambda_m^{2s}\Pi_{n_0}[(\Pi_{n_1}\Lambda_m^{-1}u_1)\cdots(\Pi_{n_{p+1}}\Lambda_m^{-1}u_{p+1})].$$

It follows that (2.2.40) holds true.

Now we are left to check that $\widetilde{M}_\ell^p \in \widetilde{\mathscr{M}}_{p+1}^{v,2s-1}(\tilde{\omega}_\ell)$ and $\widetilde{R}_\ell^p \in \widetilde{\mathscr{R}}_{p+1}^{v,2s}(\tilde{\omega}_\ell)$.

Because of cut-off function and the support property of function B in the definition of $\widetilde{M}_\ell^p$ we know that (2.1.1) holds true for $\widetilde{M}_\ell^p$ and we may assume $n_0 \sim n_{p+1}$ when estimating L^2 norm of $\Pi_{n_0}\widetilde{M}_\ell^p(\Pi_{n_1}u_1, \dots, \Pi_{n_{p+1}}u_{p+1})$. Since there is a Λ_m^{-1} following each orthogonal projector Π_{n_j}, $\quad j = 1, \dots, p+1$, we see that (1.3.4) implies (2.1.2) with $\tau = 2s - 1$ and some $v > 0$. Moreover, (2.1.15) with $\omega = \tilde{\omega}_\ell$ is satisfied by the definition of $\widetilde{M}_\ell^p$. So $\widetilde{M}_\ell^p \in \widetilde{\mathscr{M}}_{p+1}^{v,2s-1}(\tilde{\omega}_\ell)$.

Assume $\Pi_{n_0}[R(\Pi_{n_1}u_1, \dots, \Pi_{n_{p+1}}u_{p+1})]$ does not vanish. Then we have $|n_0 - n_{p+1}| \geq c(n_0 + n_{p+1})$ for some small $c > 0$ because of the cut-off function and $n_{p+1} \geq \max\{n_1, \dots, n_p\} = n'$ because of the support property of function B. Therefore if $n_0 \geq n'$, we have

$$\mu(n_0, \dots, n_{p+1}) = \left(1 + \sqrt{n'}\right)\left(1 + \min\{\sqrt{n_0}, \sqrt{n_{p+1}}\}\right),$$

$$S(n_0, \dots, n_{p+1}) = |n_0 - n_{p+1}| + \left(1 + \sqrt{n'}\right)\left(1 + \min\{\sqrt{n_0}, \sqrt{n_{p+1}}\}\right),$$

and thus

$$\frac{\mu(n_0, \dots, n_{p+1})}{S(n_0, \dots, n_{p+1})} \leq C\frac{1 + \sqrt{n'}}{\sqrt{n_0} + \sqrt{n_{p+1}} + 1 + \sqrt{n'}} \leq C\frac{\max_2(\sqrt{n_1}, \dots, \sqrt{n_{p+1}})}{1 + \sqrt{n_0} + \cdots + \sqrt{n_{p+1}}};$$

if $n_0 < n'$, we have

$$\mu(n_0, \dots, n_{p+1}) \leq \left(1 + \sqrt{n'}\right)^2, \quad S(n_0, \dots, n_{p+1}) = |n' - n_{p+1}| + \mu(n_0, \dots, n_{p+1}),$$

and thus

$$\frac{\mu(n_0, \dots, n_{p+1})}{S(n_0, \dots, n_{p+1})} \leq C\frac{1 + \sqrt{n'}}{\sqrt{n'} + \sqrt{n_{p+1}} + 1 + \sqrt{n'}} \leq C\frac{\max_2(\sqrt{n_1}, \dots, \sqrt{n_{p+1}})}{1 + \sqrt{n_0} + \cdots + \sqrt{n_{p+1}}}.$$

Now using Theorem 1.3.1 we see that (2.1.17) holds true with $\tau = 2s$ and some $v > 0$. But (2.1.24) with $\omega = \tilde{\omega}_\ell$ is satisfied according to the definition. So $\widetilde{R}_\ell^p \in \widetilde{\mathscr{R}}_{p+1}^{v,2s}(\tilde{\omega}_\ell)$. This concludes the proof of lemma. □

Summarizing the above analysis gives an end to the proof of the Proposition 2.2.1. □

In order to control the energy, let us first turn to some useful estimates in the following subsection.

2.3. *Geometric Bounds*

This subsection is a modification of Section 2.1 in [9]. We give it for the convenience of the reader. Consider the function on $\mathbb{R}^{p+2}$ depending on the parameter $m \in (0, +\infty)$, defined for $\ell = 0, \ldots, p+1$ by

$$F_m^\ell(\xi_0, \ldots, \xi_{p+1}) = \sum_{j=0}^{\ell} \sqrt{m^2 + \xi_j^2} - \sum_{j=\ell+1}^{p+1} \sqrt{m^2 + \xi_j^2}. \tag{2.3.1}$$

The main result of this subsection is the following theorem:

Theorem 2.3.1. *For any $\rho > 0$, there is a zero measure subset $\mathcal{N}$ of $\mathbb{R}_+^*$ such that for any integers $0 \le \ell \le p+1$, any $m \in \mathbb{R}_+^* - \mathcal{N}$, there are constants $c > 0$, $N_0 \in \mathbb{N}$ such that the lower bound*

$$|F_m^\ell(\lambda_{n_0}, \ldots, \lambda_{n_{p+1}})| \ge c\big(1 + \sqrt{n_0} + \sqrt{n_{p+1}}\big)^{-3-\rho}\big(1 + |\sqrt{n_0} - \sqrt{n_{p+1}}| + \sqrt{n'}\big)^{-2N_0} \tag{2.3.2}$$

holds true for any $(n_0, \ldots, n_{p+1}) \in \mathbb{N}^{p+2} - S_p^\ell$. Here λ_n are given by (1.1.1), $n' = \max\{n_1, \ldots, n_p\}$, and S_p^ℓ is defined in (2.2.18), in which we have set $\omega_\ell(j) = -1, j = 0, \ldots, \ell, \omega_\ell(j) = 1, j = \ell+1, \ldots, p+1$.

The proof of the theorem will rely on some geometric estimates that we shall deduce from results of [11]. Let us show that under the condition of Theorem 2.3.1 we have

$$\begin{aligned}|F_m^\ell(\lambda_{n_0}, \ldots, \lambda_{n_{p+1}})| \ge c(1 + \sqrt{n_0} + \sqrt{n_{p+1}})^{-3-\rho}(1 + |\sqrt{n_0} - \sqrt{n_{p+1}}|)^{-N_0} \\ \times (1 + \sqrt{n_1} + \cdots + \sqrt{n_p})^{-N_0}.\end{aligned} \tag{2.3.3}$$

Let $I \subset (0, +\infty)$ be some compact interval and define for $0 \le \ell \le p+1$ functions

$$\begin{aligned} f_\ell &: [0,1] \times [0,1]^{p+2} \times I \longrightarrow \mathbb{R} \quad (z, x_0, \ldots, x_{p+1}, y) \to f_\ell(z, x_0, \ldots, x_{p+1}, y) \\ g_\ell &: [0,1] \times [0,1]^{p} \times I \longrightarrow \mathbb{R} \quad (z, x_1, \ldots, x_{p}, y) \to g_\ell(z, x_1, \ldots, x_{p}, y) \end{aligned} \tag{2.3.4}$$

by

$$\begin{aligned} f_\ell(z, x_0, \ldots, x_{p+1}, y) &= \sum_{j=0}^{\ell} \sqrt{z^2 + y^2 x_j^2} - \sum_{j=\ell+1}^{p+1} \sqrt{z^2 + y^2 x_j^2} \\ g_\ell(z, x_1, \ldots, x_p, y) &= z\left[\sum_{j=1}^{\ell} \frac{z}{\sqrt{z^2 + y^2 x_j^2}} - \sum_{j=\ell+1}^{p} \frac{z}{\sqrt{z^2 + y^2 x_j^2}}\right] \qquad \text{when } z > 0, \\ g_\ell(0, x_1, \ldots, x_p, y) &\equiv 0. \end{aligned} \tag{2.3.5}$$

Then the graphs of f_ℓ, g_ℓ are subanalytic subsets of $[0,1]^{p+3}\times I$ and $[0,1]^{p+1}\times I$ respectively, so that f_ℓ, g_ℓ are continuous subanalytic functions (refer to Bierstone and Milman [5] for an introduction to subanalytic sets and functions). Let us consider the set Γ of points $(z,x)\in[0,1]^{p+3}$(resp. $(z,x)\in[0,1]^{p+1}$) such that $y\to f_\ell(z,x,y)$ (resp. $y\to g_\ell(z,x,y)$) vanishes identically. If $(z,x)\in\Gamma$ and $z\neq 0$, we have

$$\ell=\frac{p}{2}\quad\text{and}\quad\sum_{j\leq\ell}x_j^{2\kappa}-\sum_{j\geq\ell+1}x_j^{2\kappa}=0,\quad\forall\kappa\in\mathbb{N}^*$$

where the sum is taken respectively for $0\leq j\leq p+1$ in the case of f_ℓ and $1\leq j\leq p$ for g_ℓ. This implies that there is a bijection $\sigma:\{0,\dots,\ell\}\to\{\ell+1,\dots,p+1\}$ (resp. $\{1,\dots,\ell\}\to\{\ell+1,\dots,p\}$) such that $x_{\sigma(j)}=x_j$ for any $j=0,\dots,\ell$ (resp. $j=1,\dots,\ell$)—see for instance the proof of Lemma 5.6 in [11]. When p is even, denote by S_p the set of all bijections respectively from $\{0,\dots,\frac{p}{2}\}$ to $\{\frac{p}{2}+1,\dots,p+1\}$ and from $\{1,\dots,\frac{p}{2}\}$ to $\{\frac{p}{2},\dots,p\}$. Define for $0\leq\ell\leq p+1$

$$\begin{aligned}\rho_\ell(z,x)&\equiv z\quad\text{if }\ell\neq\frac{p}{2},\\ \rho_\ell(z,x)&=z\prod_{\sigma\in S_p}\Big[\sum_{j\leq p/2}(x_{\sigma(j)}^2-x_j^2)^2\Big]\quad\text{if }\ell=\frac{p}{2},\end{aligned}\tag{2.3.6}$$

where the sum in the above formula is taken for $j\geq 0$ (resp. $j\geq 1$) when we study f_ℓ (resp. g_ℓ). Then the set $\{\rho_\ell=0\}$ contains those points (z,x) such that $y\to f_\ell(z,x,y)$ (resp. $y\to g_\ell(z,x,y)$) vanishes identically. The following proposition is the same as Proposition 2.1.2 in [9].

Proposition 2.3.2.

(i) *There are* $\widetilde{N}\in\mathbb{N},\alpha_0>0,\delta>0,C>0,$ *such that for any* $0\leq\ell\leq p+1$, *any* $\alpha\in(0,\alpha_0)$, *any* $(z,x)\in[0,1]^{p+3}$ (*resp.* $(z,x)\in[0,1]^{p+1}$) *with* $\rho_\ell(z,x)\neq 0$, *any* $N\geq\widetilde{N}$ *the sets*

$$\begin{aligned}I_\ell^f(z,x,\alpha)&=\{y\in I;\,|f_\ell(z,x,y)|<\alpha\rho_\ell(z,x)^N\}\\ I_\ell^g(z,x,\alpha)&=\{y\in I;\,|g_\ell(z,x,y)|<\alpha\rho_\ell(z,x)^N\}\end{aligned}\tag{2.3.7}$$

have Lebesgue measure bounded from above by $C\alpha^\delta\rho_\ell(z,x)^{N\delta}$.

(ii) *For any* $N\geq\widetilde{N}$, *there is* $K\in\mathbb{N}$ *such that for any* $\alpha\in(0,\alpha_0)$, *any* $(z,x)\in[0,1]^{p+1}$, *the set* $I_\ell^g(z,x,\alpha)$ *may be written as the union of at most* K *open disjoint subintervals of* I.

We shall deduce (2.3.3) from several lemmas. Let us first introduce some notations. When p is odd or p is even and $\ell\neq\frac{p}{2}$, we set $\mathbb{N}_\ell'^p=\emptyset$. When p is even and $\ell=\frac{p}{2}$, we define

$$\begin{aligned}\mathbb{N}_\ell'^p=\{\tilde{n}=(n_1,\dots,n_p)\in\mathbb{N}^p;\ &\text{there is a bijection}\\ &\sigma:\{1,\dots,\ell\}\to\{\ell+1,\dots,p\}\text{ such that }n_{\sigma(j)}=n_j,\ j=1,\dots,\ell\}.\end{aligned}\tag{2.3.8}$$

We set also

$$\mathbb{N}_{\ell}^{p+2} = \{(n_0, \ldots, n_{p+1}) \in \mathbb{N}^{p+2}; \tilde{n} \in \mathbb{N}_{\ell}^{\prime p} \text{ and } n_0 = n_{p+1}\}. \tag{2.3.9}$$

Of course, $\mathbb{N}_{\ell}^{p+2} = \emptyset$ if p is odd or p is even and $\ell \neq \frac{p}{2}$.

We remark first that it is enough to prove (2.3.3) for those $(n_1, \ldots, n_p)$ which do not belong to $\mathbb{N}_{\ell}^{\prime p}$: actually if p is even, $\ell = \frac{p}{2}$ and $(n_1, \ldots, n_p) \in \mathbb{N}_{\ell}^{\prime p}$, we have $|F_m^{\ell}(\lambda_{n_0}, \ldots, \lambda_{n_{p+1}})| = \left|\sqrt{m^2 + \lambda_{n_0}^2} - \sqrt{m^2 + \lambda_{n_{p+1}}^2}\right|$ which is bounded from below, when m stays in some compact interval, by

$$\frac{2|n_0 - n_{p+1}|}{\sqrt{m^2 + \lambda_{n_0}^2} + \sqrt{m^2 + \lambda_{n_{p+1}}^2}} \geq \frac{c}{1 + \lambda_{n_0} + \lambda_{n_{p+1}}}$$

since from $(n_0, \ldots, n_{p+1}) \in \mathbb{N}^{p+2} - S_p^{\ell}$, we have $n_0 \neq n_{p+1}$. Consequently (2.3.3) holds true trivially. From now on, we shall always consider p-tuple $\tilde{n}$ which do not belong to $\mathbb{N}_{\ell}^{\prime p}$.

Let us define for $\ell = 1, \ldots, p$ another function on $\mathbb{R}^p$ given by

$$G_m^{\ell}(\xi_1, \ldots, \xi_p) = \sum_{j=1}^{\ell} \sqrt{m^2 + \xi_j^2} - \sum_{j=\ell+1}^{p} \sqrt{m^2 + \xi_j^2}. \tag{2.3.10}$$

Let $J \subset (0, +\infty)$ be a given compact interval. For $\alpha > 0, N_0 \in \mathbb{N}, 0 \leq \ell \leq p+1$, $n = (n_0, \ldots, n_{p+1}) \in \mathbb{N}^{p+2}$ define

$$\begin{aligned} E_J^{\ell}(n, \alpha, N_0) = \big\{m \in J; |F_m^{\ell}(\lambda_{n_0}, \ldots, \lambda_{n_{p+1}})| &< \alpha(1 + \lambda_{n_0} + \lambda_{n_{p+1}})^{-3-\rho} \\ &\times (1 + |\lambda_{n_0} - \lambda_{n_{p+1}}|)^{-N_0}(1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{-N_0}\big\}. \end{aligned} \tag{2.3.11}$$

We set also for $\beta > 0, N_1 \in \mathbb{N}^*, \tilde{n} = (n_1, \ldots, n_p) \in \mathbb{N}^p - \mathbb{N}_{\ell}^{\prime p}$

$$E_J^{\prime\ell}(\tilde{n}, \beta, N_1) = \left\{m \in J; \left|\frac{\partial G_m^{\ell}}{\partial m}(\lambda_{n_1}, \ldots, \lambda_{n_p})\right| < \beta(1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{-N_1}\right\}. \tag{2.3.12}$$

We define for $\gamma > \beta$ a subset of $\mathbb{N}^{p+2}$ by

$$\begin{aligned} S(\beta, \gamma, N_1) = \bigg\{(n_0, \ldots, n_{p+1}) \in \mathbb{N}^{p+2} - \mathbb{N}_{\ell}^{p+2} : \lambda_{n_0} &< \frac{\gamma}{3\beta}(1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{N_1} \\ \text{or } \lambda_{n_{p+1}} &< \frac{\gamma}{3\beta}(1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{N_1}\bigg\}. \end{aligned} \tag{2.3.13}$$

Lemma 2.3.3. *Let $\widetilde{N}, \delta, \alpha_0$ be the constants defined in the statement of Proposition 2.3.2. There are constants $C_1 > 0, M \in \mathbb{N}^*$ such that for any $\beta \in (0, \alpha_0)$, any $N_1 \in \mathbb{N}$ with $N_1 > M\widetilde{N}$ and $N_1 > \frac{2pM}{\delta}$, one has*

$$meas\left[\bigcup_{\tilde{n} \in \mathbb{N}^p - \mathbb{N}_{\ell}^{\prime p}} E_J^{\prime\ell}(\tilde{n}, \beta, N_1)\right] \leq C_1 \beta^{\delta}. \tag{2.3.14}$$

Proof. Set $y = \frac{1}{m}$ and

$$z = \left(1 + \sum_{j=1}^{p} \lambda_{n_j}\right)^{-1}, \quad x_j = \lambda_{n_j} z, \quad j = 1, \ldots, p.$$

Denote by X the set of points $(z, x) \in [0, 1]^{p+1}$ of the preceding form for $(n_1, \ldots, n_p)$ describing $\mathbb{N}^p$. When p is even and $\ell = p/2$, let X'^p_ℓ be the imagine of $\mathbb{N}'^p_\ell$ defined by (2.3.8) under the map $\tilde{n} \to (z, x)$. Using Definition 2.3.6, we see that there are constants $M > 0$, $C > 0$, depending only on p, such that for $0 \le \ell \le p+1$

$$\forall (z, x) \in X - X'^p_\ell, \quad z^M \le \rho_\ell(z, x) \le Cz \tag{2.3.15}$$

since, when $\ell = \frac{p}{2}$ and $(n_1, \ldots, n_p) \notin \mathbb{N}'^p_\ell$, $\sum_{j=1}^{\frac{p}{2}} (\lambda^2_{n_{\sigma(j)}} - \lambda^2_{n_j})^2 \ge 1$, by the definition of λ_{n_j}. Remark that with the above notations

$$\frac{\partial G^\ell_m}{\partial m}(\lambda_{n_1}, \ldots, \lambda_{n_p}) = \sum_{j=1}^{\ell} \frac{m}{\sqrt{m^2 + \lambda^2_{n_j}}} - \sum_{j=\ell+1}^{p} \frac{m}{\sqrt{m^2 + \lambda^2_{n_j}}} = \frac{1}{z} g_\ell(z, x_1, \ldots, x_p, y).$$

Then if $I = \{m^{-1}; m \in J\}$, we see that $m \in E'^\ell_J(\tilde{n}, \beta, N_1)$ for $n \notin \mathbb{N}'^p_\ell$ if and only if $y = \frac{1}{m}$ satisfies

$$|g_\ell(z, x_1, \ldots, x_p, y)| < \beta z^{N_1+1} \le \beta \rho_\ell(z, x)^{\frac{1}{M}(N_1+1)} \tag{2.3.16}$$

using (2.3.15). Applying Proposition 2.3.2(i), we see that for any fixed value of $(z, x) \in X - X'^p_\ell$, the measure of those y such that (2.3.16) holds true is bounded from above by

$$C\beta^\delta \rho_\ell(z, x)^{\frac{N_1+1}{M}\delta} \le C\beta^\delta z^{\frac{N_1+1}{M}\delta}$$

if we assume $N_1 \ge M\widetilde{N}$ and $\beta \in (0, \alpha_0)$. Consequently, we get with a constant C' depending only on J,

$$\begin{aligned} meas(E'^\ell_J(n', \beta, N_1)) &\le C'\beta^\delta (1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{-\frac{N_1+1}{M}\delta} \\ &\le C'\beta^\delta (1 + n_1 + \cdots + n_p)^{-\frac{N_1+1}{2M}\delta}. \end{aligned}$$

Inequality (2.3.14) follows from this estimate and the assumption on N_1. □

Lemma 2.3.4. *Let $\widetilde{N}, \delta, \alpha_0$ be the constants defined in the statement of Proposition 2.3.2. There are constants $M \in \mathbb{N}^*, \theta > 1, C_2 > 0$ such that for any $N_0, N_1 \in \mathbb{N}^*$ satisfying $N_0 > \widetilde{N}MN_1$ and $N_0\delta > 2(p+2)MN_1$, any $0 < \beta < \gamma$ with $\frac{\gamma}{\beta} > \theta$, any $\alpha > 0$ satisfying $\alpha\left(\frac{\beta}{2\gamma}\right)^{-\frac{N_0}{N_1}} < \alpha_0$, one has*

$$meas\left[\bigcup_{n \in S(\beta, \gamma, N_1)} E^\ell_J(n, \alpha, N_0)\right] \le C_2 \alpha^\delta \left(\frac{\beta}{2\gamma}\right)^{-\frac{N_0}{N_1}\delta}. \tag{2.3.17}$$

Proof. We first remark that if $\lambda_{n_0} + \lambda_{n_{p+1}} > \frac{\gamma}{\beta}(1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{N_1}$ and $n \in S(\beta, \gamma, N_1)$, then either

$$\lambda_{n_0} \geq \frac{2\gamma}{3\beta}(1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{N_1} \quad \text{or} \quad \lambda_{n_{p+1}} \geq \frac{2\gamma}{3\beta}(1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{N_1},$$

which implies that

$$|F_m^\ell(\lambda_{n_0}, \ldots, \lambda_{n_{p+1}})| \geq c\frac{\gamma}{\beta}(1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{N_1}$$

for some constant $c > 0$ depending only on p and J, if $\frac{\gamma}{\beta} > \theta$ large enough. Consequently, if $\alpha < \alpha_0$ small enough relatively to c, we see that we have in this case $E_J^\ell(n, \alpha, N_0) = \emptyset$ when $n \in S(\beta, \gamma, N_1)$. We may therefore consider only indices n such that

$$n \in S(\beta, \gamma, N_1) \quad \text{and} \quad \lambda_{n_0} + \lambda_{n_{p+1}} \leq \frac{\gamma}{\beta}(1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{N_1}.$$

Consequently, for $m \in E_J^\ell(n, \alpha, N_0)$ and $n \in S(\beta, \gamma, N_1)$, we have

$$\begin{aligned} |F_m^\ell(\lambda_{n_0}, \ldots, \lambda_{n_{p+1}})| &\leq \alpha(1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{-N_0} \\ &\leq \alpha\left(\frac{\beta}{2\gamma}\right)^{-\frac{N_0}{N_1}}(1 + \lambda_{n_0} + \cdots + \lambda_{n_{p+1}})^{-\frac{N_0}{N_1}}. \end{aligned} \tag{2.3.18}$$

Define for $n \in \mathbb{N}^{p+2}$

$$z = \left(1 + \sum_{j=0}^{p+1} \lambda_{n_j}\right)^{-1}, \quad x_j = \lambda_{n_j} z, \quad j = 0, \ldots, p+1. \tag{2.3.19}$$

Denote by $X \subset [0, 1]^{p+3}$ the set of points (z, x) of the preceding form, and let X_ℓ^p be the imagine of the set $\mathbb{N}_\ell^{p+2}$ defined by (2.3.9) under the map $n \to (z, x)$. By (2.3.6) we have again

$$\forall (z, x) \in X - X_\ell^p, \quad z^M \leq \rho_\ell(z, x) \leq Cz$$

for some large enough M, depending only on p. Moreover

$$F_m^\ell(\lambda_{n_0}, \ldots, \lambda_{n_{p+1}}) = \frac{m}{z} f_\ell(z, x_0, \ldots, x_{p+1}, y)$$

and (2.3.18) implies that if $n \in S(\beta, \gamma, N_1)$ and $m \in E_J^\ell(n, \alpha, N_0)$, then y satisfies

$$\begin{aligned} |f_\ell(z, x_0, \ldots, x_{p+1}, y)| &\leq C\alpha\left(\frac{\beta}{2\gamma}\right)^{-\frac{N_0}{N_1}} z^{1+\frac{N_0}{N_1}} \\ &\leq C\alpha\left(\frac{\beta}{2\gamma}\right)^{-\frac{N_0}{N_1}} \rho_\ell(z, x)^{\frac{1}{M}(1+\frac{N_0}{N_1})} \end{aligned} \tag{2.3.20}$$

We assume that α, N_0, N_1 satisfy the conditions of the statement of the lemma. Then by (i) of Proposition 2.3.2 we get that the measure of those $y \in J$ satisfying (2.3.20) is bounded from above by

$$C\left[\alpha\left(\frac{\beta}{2\gamma}\right)^{-\frac{N_0}{N_1}}\right]^{\delta} z^{\frac{\delta}{M}(1+\frac{N_0}{N_1})}$$

for some constant C, independent of $N_0, N_1, \alpha, \beta, \gamma$. Consequently the measure of $E_J^{\ell}(n, \alpha, N_0)$ is bounded from above when $n \in S(\beta, \gamma, N_1)$ by

$$\begin{aligned} &C\left[\alpha\left(\frac{\beta}{2\gamma}\right)^{-\frac{N_0}{N_1}}\right]^{\delta}\left(1+\lambda_{n_0}+\cdots+\lambda_{n_{p+1}}\right)^{-\frac{\delta}{M}(1+\frac{N_0}{N_1})} \\ &\quad \leq C'\left[\alpha\left(\frac{\beta}{2\gamma}\right)^{-\frac{N_0}{N_1}}\right]^{\delta}\left(1+n_0+\cdots+n_{p+1}\right)^{-\frac{\delta}{2M}(1+\frac{N_0}{N_1})} \end{aligned}$$

for another constant C' depending on J. The conclusion of the lemma follows by summation, using that $\frac{\delta}{M}(1+\frac{N_0}{N_1}) > 2(p+2)$. □

Proof of Theorem 2.3.1. We fix N_0, N_1 satisfying the conditions stated in Lemmas 2.3.3 and 2.3.4, and such that $N_0 > 2p + N_1$. We write when $n \notin S(\beta, \gamma, N_1)$, $0 \leq \ell \leq p+1$,

$$E_J^{\ell}(n, \alpha, N_0) \subset [E_J^{\ell}(n, \alpha, N_0) \cap E_J'^{\ell}(\tilde{n}, \beta, N_1)] \cup [E_J^{\ell}(n, \alpha, N_0) \cap E_J'^{\ell}(\tilde{n}, \beta, N_1)^c]$$

and estimate, using that we reduced ourselves to those $\tilde{n} \notin \mathbb{N}_{\ell}'^{p}$

$$\begin{aligned} meas\left[\bigcup_{n;\tilde{n}\notin\mathbb{N}_{\ell}'^{p}} E_J^{\ell}(n, \alpha, N_0)\right] &\leq meas\left[\bigcup_{n\in S(\beta,\gamma,N_1)} E_J^{\ell}(n, \alpha, N_0)\right] + meas\left[\bigcup_{\tilde{n}\notin\mathbb{N}_{\ell}'^{p}} E_J'^{\ell}(\tilde{n}, \beta, N_1)\right] \\ &\quad + meas\left[\bigcup_{n\in S(\beta,\gamma,N_0)^c-\mathbb{N}_{\ell}^{p+2}} E_J^{\ell}(n, \alpha, N_0) \cap E_J'^{\ell}(\tilde{n}, \beta, N_1)^c\right]. \end{aligned} \tag{2.3.21}$$

Let us bound the measure of $E_J^{\ell}(n, \alpha, N_0) \cap E_J'^{\ell}(\tilde{n}, \beta, N_1)^c$ for $n \in S(\beta, \gamma, N_0)^c - \mathbb{N}_{\ell}^{p+2}$. If m belongs to that set, the inequality in (2.3.11) holds true. Remark that we may assume $\ell \leq p$: if $\ell = p+1$, $|F_m^{\ell}(\lambda_{n_0}, \ldots, \lambda_{n_{p+1}})| \geq c(1 + \lambda_{n_0} + \lambda_{n_{p+1}})$ for some $c > 0$, which is not compatible with (2.3.11) for $\alpha < \alpha_0$ small enough. Let us write (2.3.11) as

$$\begin{aligned} &|\lambda_{n_0} - \lambda_{n_{p+1}} + \widetilde{G}_m(\lambda_{n_0}, \ldots, \lambda_{n_{p+1}})| \\ &\quad < \alpha(1+\lambda_{n_0}+\lambda_{n_{p+1}})^{-3-\rho}(1+|\lambda_{n_0}-\lambda_{n_{p+1}}|)^{-N_0}(1+\lambda_{n_1}+\cdots+\lambda_{n_p})^{-N_0} \end{aligned} \tag{2.3.22}$$

with, using notation (2.3.10)

$$\begin{aligned} \widetilde{G}_m(\lambda_{n_0}, \ldots, \lambda_{n_{p+1}}) &= G_m(\lambda_{n_1}, \ldots, \lambda_{n_p}) + R_m(\lambda_{n_0}, \lambda_{n_{p+1}}) \\ R_m(\lambda_{n_0}, \lambda_{n_{p+1}}) &= \left(\sqrt{m^2+\lambda_{n_0}^2} - \lambda_{n_0}\right) - \left(\sqrt{m^2+\lambda_{n_{p+1}}^2} - \lambda_{n_{p+1}}\right). \end{aligned} \tag{2.3.23}$$

Since $n \in S(\beta, \gamma, N_1)^c$, we have by (2.3.13)

$$\lambda_{n_0} \geq \frac{\gamma}{3\beta}(1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{N_1}, \quad \lambda_{n_{p+1}} \geq \frac{\gamma}{3\beta}(1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{N_1}. \tag{2.3.24}$$

Consequently there is a constant $C > 0$, depending only on J, such that

$$\left|\frac{\partial R_m}{\partial m}(\lambda_{n_0}, \lambda_{n_{p+1}})\right| \leq C\frac{\beta}{\gamma}(1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{-N_1}.$$

If γ is large enough and $m \in E_J'^{\ell}(\tilde{n}, \beta, N_1)^c$, we deduce from (2.3.12) that

$$\left|\frac{\partial \widetilde{G}_m}{\partial m}(\lambda_{n_0}, \ldots, \lambda_{n_{p+1}})\right| \geq \frac{\beta}{2}(1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{-N_1}. \tag{2.3.25}$$

By (ii) of Proposition 2.3.2, we know that there is $K \in \mathbb{N}$, independent of α, β, γ such that the set $J - E_J'^{\ell}(\tilde{n}, \beta, N_1)$ is the union of at most K disjoint intervals $J_j(\tilde{n}, \beta, N_1)$, $1 \leq j \leq K$. Consequently, we have

$$E_J^{\ell}(n, \alpha, N_0) \cap (E_J'^{\ell}(\tilde{n}, \beta, N_1))^c \subset \bigcup_{j=1}^{K}\{m \in J_j(\tilde{n}, \beta, N_1); \text{(2.3.22) holds true}\}, \tag{2.3.26}$$

and on each interval $J_j(n', \beta, N_1)$, (2.3.25) holds true. We may on each such interval perform in the characteristic function of (2.3.22) the change of variable of integration given by $m \to \widetilde{G}_m(\lambda_{n_0}, \ldots, \lambda_{n_{p+1}})$. Because of (2.3.25) this allows us to estimate the measure of (2.3.26) by

$$\begin{aligned}
&K\frac{2}{\beta}\alpha(1 + \lambda_{n_0} + \lambda_{n_{p+1}})^{-3-\rho}(1 + |\lambda_{n_0} - \lambda_{n_{p+1}}|)^{-N_0}(1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{-N_0+N_1} \\
&\quad \leq CK\frac{2}{\beta}\alpha(1 + n_0 + n_{p+1})^{-\frac{1}{2}(3+\rho)}\big(1 + |\sqrt{n_0} - \sqrt{n_{p+1}}|\big)^{-N_0}(1 + n_1 + \cdots + n_p)^{-\frac{1}{2}(N_0-N_1)}
\end{aligned}$$

Summing in $n_0, \ldots, n_{p+1}$, we see that since $N_0 > 2p + N_1$, the last term in (2.3.21) is bounded from above by $C_3\frac{\alpha}{\beta}$ with C_3 independent of α, β, γ. By Lemmas 2.3.3 and 2.3.4, we may thus bound (2.3.21) by

$$C_2\alpha^{\delta}\left(\frac{\beta}{2\gamma}\right)^{-\frac{N_0}{N_1}\delta} + C_1\beta^{\delta} + C_3\frac{\alpha}{\beta}$$

if α, β are small enough, γ is large enough and $\alpha\left(\frac{\beta}{\gamma}\right)^{-\frac{N_0}{N_1}}$ is small enough. If we take $\beta = \alpha^{\sigma}$, $\gamma = \alpha^{-\sigma}$ with $\sigma > 0$ small enough, and $\alpha \ll 1$, we finally get for some $\delta' > 0$,

$$meas\left[\bigcup_{n;\tilde{n}\notin\mathbb{N}_{\ell}'^{p}} E_J^{\ell}(n, \alpha, N_0)\right] \leq C\alpha^{\delta'} \to 0 \quad \text{if } \alpha \to 0^+.$$

This implies that in this case the set of those $m \in J$ for which (2.3.3) does not hold true for any $c > 0$ is of zero measure. This concludes the proof. □

We will need a consequence of Theorem 2.3.1:

Proposition 2.3.5. *For any $\rho > 0$, there is a zero measure subset $\mathcal{N}$ of $\mathbb{R}_+^*$ such that for any integers $0 \le \ell \le p+1$, any $m \in \mathbb{R}_+^* - \mathcal{N}$, there are constants $c > 0, N_0 \in \mathbb{N}$ such that the lower bound*

$$|F_m^\ell(\lambda_{n_0}, \dots, \lambda_{n_{p+1}})| \ge c(1 + \sqrt{n_0} + \sqrt{n_{p+1}})^{-3-\rho}(1 + \sqrt{n'})^{-2N_0} \frac{\mu(n_0, \dots, n_{p+1})^{2N_0}}{S(n_0, \dots, n_{p+1})^{2N_0}} \tag{2.3.27}$$

holds true for any $(n_0, \dots, n_{p+1}) \in \mathbb{N}^{p+2} - S_p^\ell$ with $n_0 \sim n_{p+1}$ and $n_{p+1} \ge n'$. Here λ_n, n', S_p^ℓ are the same as those in Theorem 2.3.1.

Proof. By Theorem 2.3.1 we know (2.3.2) holds true under the conditions of the proposition. Since we assume $n_0 \sim n_{p+1}$ and $n_{p+1} \ge n'$, we have by (1.3.2) and (1.3.3)

$$\begin{aligned} \mu(n_0, \dots, n_{p+1}) &\sim \big(1 + \sqrt{n_{p+1}}\big)\big(1 + \sqrt{n'}\big), \\ S(n_0, \dots, n_{p+1}) &\sim |n_0 - n_{p+1}| + \big(1 + \sqrt{n_{p+1}}\big)\big(1 + \sqrt{n'}\big) \\ &\sim \big(1 + \sqrt{n_{p+1}}\big)\big(1 + |\sqrt{n_0} - \sqrt{n_{p+1}}| + \sqrt{n'}\big). \end{aligned} \tag{2.3.28}$$

Therefore we deduce from (2.3.2)

$$\begin{aligned} |F_m^\ell(\lambda_{n_0}, \dots, \lambda_{n_{p+1}})| &\ge c(1 + \sqrt{n_0} + \sqrt{n_{p+1}})^{-3-\rho} \frac{(1 + \sqrt{n_{p+1}})^{2N_0}}{S(n_0, \dots, n_{p+1})^{2N_0}} \\ &\ge c(1 + \sqrt{n_0} + \sqrt{n_{p+1}})^{-3-\rho}(1 + \sqrt{n'})^{-2N_0} \frac{\mu(n_0, \dots, n_{p+1})^{2N_0}}{S(n_0, \dots, n_{p+1})^{2N_0}}. \end{aligned}$$

This concludes the proof of the proposition. □

In the following subsection, we shall also use a simpler version of Theorem 2.3.1. Let us introduce some notations. For $m \in \mathbb{R}_+^*$, $\xi_j \in \mathbb{R}$, $j = 0, \dots, p+1$, $e = (e_0, \dots, e_{p+1}) \in \{-1, 1\}^{p+2}$, define

$$\widetilde{F}_m^{(e)}(\xi_0, \dots, \xi_{p+1}) = \sum_{j=0}^{p+1} e_j \sqrt{m^2 + \xi_j^2}. \tag{2.3.29}$$

When p is even and $\#\{j; e_j = 1\} = \frac{p}{2} + 1$, denote by $N^{(e)}$ the set of all $(n_0, \dots, n_{p+1}) \in \mathbb{N}^{p+2}$ such that there is a bijection σ from $\{j; 0 \le j \le p+1, e_j = 1\}$ to $\{j; 0 \le j \le p+1, e_j = -1\}$ so that for any j in the first set $n_j = n_{\sigma(j)}$. In the other cases, set $N^{(e)} = \emptyset$.

Proposition 2.3.6. *There is a zero measure subset $\mathcal{N}$ of $\mathbb{R}_+^*$ and for any $m \in \mathbb{R}_+^* - \mathcal{N}$, there are constants $c > 0, N_0 \in \mathbb{N}$ such that for any $(n_0, \dots, n_{p+1}) \in \mathbb{N}^{p+2} - N^{(e)}$ one has*

$$\big|\widetilde{F}_m^{(e)}(\lambda_{n_0}, \dots, \lambda_{n_{p+1}})\big| \ge c\big(1 + \sqrt{n_0} + \dots + \sqrt{n_{p+1}}\big)^{-N_0}. \tag{2.3.30}$$

Moreover, if $e_0 e_{p+1} = 1$, *one has the inequality*

$$|\widetilde{F}_m^{(e)}(\lambda_{n_0}, \dots, \lambda_{n_{p+1}})| \geq c(1 + \sqrt{n_0} + \sqrt{n_{p+1}})(1 + \sqrt{n_1} + \cdots + \sqrt{n_p})^{-N_0}. \quad (2.3.31)$$

Proof. With the reasoning as in the proof of Proposition 2.1.5 in [9], we get just by replacing $(n_0, \dots, n_{p+1})$ with $(\lambda_0, \dots, \lambda_{p+1})$

$$|\widetilde{F}_m^{(e)}(\lambda_{n_0}, \dots, \lambda_{n_{p+1}})| \geq c(1 + \lambda_{n_0} + \cdots + \lambda_{n_{p+1}})^{-N_0}$$

and

$$|\widetilde{F}_m^{(e)}(\lambda_{n_0}, \dots, \lambda_{n_{p+1}})| \geq c(1 + \lambda_{n_0} + \lambda_{n_{p+1}})(1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{-N_0}$$

when $e_0 e_{p+1} = 1$. This concludes the proof of the proposition by noting (1.1.1). □

2.4. *Energy Control and the Proof of the Main Theorem*

We shall use the results of Subsection 2.3 to control the energy. When $M(u_1, \dots, u_{p+1})$ is a $p+1$-linear form, let us define for $0 \leq \ell \leq p+1$,

$$\begin{aligned} L_\ell^-(M)(u_1, \dots, u_{p+1}) = -\Lambda_m M(u_1, \dots, u_{p+1}) \\ - \sum_{j=1}^{\ell} M(u_1, \dots, \Lambda_m u_j, \dots, u_{p+1}) \\ + \sum_{j=\ell+1}^{p+1} M(u_1, \dots, \Lambda_m u_j, \dots, u_{p+1}) \end{aligned} \quad (2.4.1)$$

and

$$\begin{aligned} &L_\ell^+(M)(u_1, \dots, u_{p+1}) \\ &= -\Lambda_m M(u_1, \dots, u_{p+1}) - \sum_{j=1}^{\ell} M(u_1, \dots, \Lambda_m u_j, \dots, u_{p+1}) \\ &\quad + \sum_{j=\ell+1}^{p} M(u_1, \dots, \Lambda_m u_j, \dots, u_{p+1}) - M(u_1, \dots, u_p, \Lambda_m u_{p+1}). \end{aligned} \quad (2.4.2)$$

We shall need the following lemma:

Lemma 2.4.1. *For any* $\rho > 0$, *let* $\mathcal{N}$ *be the zero measure subset of* $\mathbb{R}_+^*$ *defined by taking the union of the zero measure subsets defined in Propositions* 2.3.5 *and* 2.3.6, *and fix* $m \in \mathbb{R}_+^* - \mathcal{N}$. *Let* $\omega_\ell, \tilde{\omega}_\ell$ *be defined in the statement of Proposition* 2.2.1. *There is a* $\bar{\nu} \in \mathbb{N}$ *such that the following statements hold true for any large enough integer s, any integer p with* $\kappa \leq p \leq 2\kappa - 1$, *any integer* ℓ *with* $0 \leq \ell \leq p$:

- *Let* ϵ, κ *be parameters in* (1.2.1) *and* $\theta \in (0, 1)$ *a constant to be chosen later. Let* $M_\ell^p \in \widetilde{\mathscr{M}}_{p+1}^{\nu, 2s-a}(\omega_\ell)$ *with* $a = 2$ *if* $d \geq 2$ *and* $a = \frac{13}{6} - \varsigma$ *for any* $\varsigma \in (0, 1)$ *if*

$d = 1$ and $\widetilde{M}_\ell^p \in \widetilde{\mathscr{M}}_{p+1}^{\nu,2s-1}(\tilde{\omega}_\ell)$. Define

$$M_\ell^{p,\epsilon}(u_1, \dots, u_{p+1}) = \sum_{n_0} \sum_{n_{p+1}} \mathbf{1}_{\{\sqrt{n_0}+\sqrt{n_{p+1}}<\epsilon^{-\theta\kappa}\}} \Pi_{n_0} M_\ell^p(u_1, \dots, u_p, \Pi_{n_{p+1}} u_{p+1}). \tag{2.4.3}$$

Then there are $\underline{M}_\ell^{p,\epsilon} \in \widetilde{\mathscr{M}}_{p+1}^{\nu+\bar{\nu},2s-1}(\omega_\ell)$ and $\underline{M}_\ell^p \in \widetilde{\mathscr{M}}_{p+1}^{\nu,2s-2}(\tilde{\omega}_\ell)$ satisfying

$$\begin{aligned} L_\ell^-(\underline{M}_\ell^{p,\epsilon})(u_1, \dots, u_{p+1}) &= M_\ell^{p,\epsilon}(u_1, \dots, u_{p+1}), \\ L_\ell^+(\underline{M}_\ell^p)(u_1, \dots, u_{p+1}) &= \widetilde{M}_\ell^p(u_1, \dots, u_{p+1}) \end{aligned} \tag{2.4.4}$$

with the estimate for all $N \geq \bar{\nu}$,

$$\begin{aligned} \|\underline{M}_\ell^{p,\epsilon}\|_{\mathscr{M}_{p+1,N}^{\nu+\bar{\nu},2s-1}} &\leq C\epsilon^{-(4-a+\rho)\theta\kappa} \|M_\ell^p\|_{\mathscr{M}_{p+1,N}^{\nu,2s-a}}, \\ \|\underline{M}_\ell^p\|_{\mathscr{M}_{p+1,N}^{\nu+\bar{\nu},2s-2}} &\leq C\|\widetilde{M}_\ell^p\|_{\mathscr{M}_{p+1,N}^{\nu,2s-1}}, \end{aligned} \tag{2.4.5}$$

where $\|\cdot\|_{\mathscr{M}_{p+1,N}^{\nu,\tau}}$ is defined in the statement of Definition 2.1.1.

- *Let $R_\ell^p \in \widetilde{R}_{p+1}^{\nu,2s}(\omega_\ell)$, $\widetilde{R}_\ell^p \in \widetilde{R}_{p+1}^{\nu,2s}(\tilde{\omega}_\ell)$. Then there are $\underline{R}_\ell^p \in \widetilde{\mathscr{R}}_{p+1}^{\nu+\bar{\nu},2s}(\omega_\ell)$ and $\underline{R}_\ell'^p \in \widetilde{\mathscr{R}}_{p+1}^{\nu+\bar{\nu},2s}(\tilde{\omega}_\ell)$ such that*

$$\begin{aligned} L_\ell^-(\underline{R}_\ell^p)(u_1, \dots, u_{p+1}) &= R_\ell^p(u_1, \dots, u_{p+1}), \\ L_\ell^+(\underline{R}_\ell'^p)(u_1, \dots, u_{p+1}) &= \widetilde{R}_\ell^p(u_1, \dots, u_{p+1}). \end{aligned} \tag{2.4.6}$$

Proof. (i) We substitute in (2.4.4) $\Pi_{n_j} u_j$ to u_j, $j = 1, \dots, p+1$, and compose on the left with Π_{n_0}. According to (2.4.1), equalities in (2.4.4) may be written

$$-F_m^\ell(\lambda_{n_0}, \dots, \lambda_{n_{p+1}})\Pi_{n_0}\underline{M}_\ell^{p,\epsilon}(\Pi_{n_1}u_1, \dots, \Pi_{n_{p+1}}u_{p+1}) = \Pi_{n_0} M_\ell^{p,\epsilon}(\Pi_{n_1}u_1, \dots, \Pi_{n_{p+1}}u_{p+1}), \tag{2.4.7}$$

$$\widetilde{F}_m^{(e)}(\lambda_{n_0}, \dots, \lambda_{n_{p+1}})\Pi_{n_0}\underline{M}_\ell^p(\Pi_{n_1}u_1, \dots, \Pi_{n_{p+1}}u_{p+1}) = \Pi_{n_0} \widetilde{M}_\ell^p(\Pi_{n_1}u_1, \dots, \Pi_{n_{p+1}}u_{p+1}), \tag{2.4.8}$$

where F_m^ℓ is defined by (2.3.1) and $\widetilde{F}_m^{(e)}$ is defined by (2.3.29) with $e_0 = \cdots = e_\ell = e_{p+1} = -1$, $e_{\ell+1} = \cdots = e_p = 1$.

When considering (2.4.7), we may assume $n_0 \sim n_{p+1}$, $n_{p+1} \geq n'$ and $(n_0, \dots, n_{p+1}) \notin S_p^\ell$ if the right hand side of (2.4.7) is non-zero since we have (2.1.1) and (2.1.15) for $M_\ell^{p,\epsilon}$. Here S_p^ℓ is the same as that in Proposition 2.3.5. Thus the assumptions concerning $(n_0, \dots, n_{p+1})$ in Proposition 2.3.5 hold true. We deduce from (2.3.27) and the condition $\sqrt{n_0} + \sqrt{n_{p+1}} < \epsilon^{-\theta\kappa}$ that

$$\begin{aligned} &|F_m^\ell(\lambda_{n_0}, \dots, \lambda_{n_{p+1}})|^{-1} \\ &\quad \leq C(1 + \sqrt{n_0} + \sqrt{n_{p+1}})^{3+\rho}(1 + \sqrt{n'})^{2N_0} \frac{S(n_0, \dots, n_{p+1})^{2N_0}}{\mu(n_0, \dots, n_{p+1})^{2N_0}} \\ &\quad \leq C\epsilon^{-(4-a+\rho)\theta\kappa}(1 + \sqrt{n_0} + \sqrt{n_{p+1}})^{a-1}(1 + \sqrt{n'})^{2N_0} \frac{S(n_0, \dots, n_{p+1})^{2N_0}}{\mu(n_0, \dots, n_{p+1})^{2N_0}} \end{aligned} \tag{2.4.9}$$

for any $\rho > 0$. Therefore if we define

$$\underline{M}_\ell^{p,\epsilon}(u_1,\dots,u_{p+1}) = -\sum_{\substack{n\notin S_p^\ell \\ n_0\sim n_{p+1},\, n_{p+1}\geq n'}} F_m^\ell(\lambda_{n_0},\dots,\lambda_{n_{p+1}})^{-1}\Pi_{n_0} M_\ell^{p,\epsilon}(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1}), \tag{2.4.10}$$

we obtain according to (2.4.9) and (2.1.2) that $\underline{M}_\ell^{p,\epsilon} \in \widetilde{\mathscr{M}}_{p+1}^{\nu+\bar\nu,2s-1}(\omega_\ell)$ with the first estimate in (2.4.5) with $\bar\nu = 2N_0$.

When considering (2.4.8), we may assume $(n_0,\dots,n_{p+1}) \notin N^{(e)}$ defined after (2.3.29). Actually, because of (2.1.15), we cannot find a bijection σ from $\{0,\dots,\ell,p+1\}$ to $\{\ell+1,\dots,p\}$ such that $n_j = n_{\sigma(j)}$, $j = 0,\dots,\ell,p+1$ if the right hand side of (2.4.8) is non zero. Consequently, we may use lower bound (2.3.31). If we define $\underline{M}_\ell^p$ dividing in (2.4.8) by $\widetilde{F}_m^{(e)}$, we thus see that we get an element of $\underline{M}_\ell^p \in \widetilde{\mathscr{M}}_{p+1}^{\nu+\bar\nu,2s-2}(\tilde\omega_\ell)$ for some $\bar\nu$. This completes the proof of (2.4.4) and (2.4.5).

(ii) We deduce again from (2.4.6)

$$-F_m^\ell(\lambda_{n_0},\dots,\lambda_{n_{p+1}})\Pi_{n_0}\underline{R}_\ell^p(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1}) = \Pi_{n_0}R_\ell^p(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1}), \tag{2.4.11}$$

$$\widetilde{F}_m^{(e)}(\lambda_{n_0},\dots,\lambda_{n_{p+1}})\Pi_{n_0}\underline{R}_\ell'^p(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1}) = \Pi_{n_0}\widetilde{R}_\ell^p(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1}), \tag{2.4.12}$$

where F_m^ℓ and $\widetilde{F}_m^{(e)}$ are the same as in (2.4.7) and (2.4.8). Since $R_\ell^p \in \widetilde{R}_{p+1}^{\nu,2s}(\omega_\ell)$ and thus (2.1.24) implies the right hand side of (2.4.11) vanishes if $(n_0,\dots,n_{p+1}) \in S_p^\ell$, where S_p^ℓ is defined in (2.2.18), we may assume $(n_0,\dots,n_{p+1}) \notin S_p^\ell$. Consequently, the condition of Theorem 2.3.1 is satisfied and we have by (2.3.2)

$$|F_m^\ell(\lambda_{n_0},\dots,\lambda_{n_{p+1}})|^{-1} \leq C(1+\sqrt{n_0}+\sqrt{n_1}+\cdots+\sqrt{n_{p+1}})^{2N_0+4}.$$

We then get an element of $\underline{R}_\ell^p \in \widetilde{\mathscr{R}}_{p+1}^{\nu+\bar\nu,2s}(\omega_\ell)$ dividing in (2.4.11) by $-F_m^\ell$ with $\bar\nu = 2N_0+4$. Since $\widetilde{R}_\ell^p \in \widetilde{R}_{p+1}^{\nu,2s}(\tilde\omega_\ell)$, we see that the right hand side of (2.4.12) vanishes if $(n_0,\dots,n_{p+1}) \in \widetilde{S}_p^\ell$, where $\widetilde{S}_p^\ell$ is defined in (2.2.41). This implies that we may assume $(n_0,\dots,n_{p+1}) \notin N^{(e)}$ which is defined after (2.3.29) with $e_0 = \cdots = e_\ell = e_{p+1} = -1$, $e_{\ell+1} = \cdots = e_p = 1$. Thus the condition of Proposition 2.3.6 is satisfied and we have

$$|\widetilde{F}_m^{(e)}(\lambda_{n_0},\dots,\lambda_{n_{p+1}})|^{-1} \leq C(1+\sqrt{n_0}+\cdots+\sqrt{n_{p+1}})^{N_0}.$$

This allows us to get an element $\underline{R}_\ell'^p \in \widetilde{\mathscr{R}}_{p+1}^{\nu+\bar\nu,2s}(\tilde\omega_\ell)$ for some $\bar\nu$ by dividing by $\widetilde{F}_m^{(e)}$ in (2.4.12). This concludes the proof. □

Proposition 2.4.2. *Let $\rho > 0$ be any positive number and $\mathscr{N}$ the zero measure subset of $\mathbb{R}_+^*$ defined in Lemma* 2.4.1, *and fix $m \in \mathbb{R}_+^* - \mathscr{N}$. Let Θ_s be defined in* (2.2.5). *There are for any large enough integer s, a map Θ_s^1, sending $\mathscr{H}^s(\mathbb{R}^d) \times \left(0, \frac{1}{2}\right)$ to $\mathbb{R}$, and maps $\Theta_s^2, \Theta_s^3, \Theta_s^4$ sending $\mathscr{H}^s(\mathbb{R}^d)$ to $\mathbb{R}$ such that there is a constant $C_s > 0$ and for any $u \in$*

$\mathcal{H}^s(\mathbb{R}^d)$ with $\|u\|_{\mathcal{H}^s} \leq 1$ and for ϵ, κ, θ the same as in Lemma 2.4.1 with $\epsilon \in (0, \frac{1}{2})$, one has

$$|\Theta_s^1(u, \epsilon)| \leq C_s \epsilon^{-(4-a+\rho)\theta\kappa} \|u\|_{\mathcal{H}^s}^{\kappa+2},$$
$$\left(a = 2 \text{ if } d \geq 2 \text{ and } a = \frac{13}{6} - \varsigma \text{ for any } \varsigma \in (0,1) \text{ if } d = 1\right),$$
$$|\Theta_s^2(u)|, |\Theta_s^3(u)|, |\Theta_s^4(u)| \leq C_s \|u\|_{\mathcal{H}^s}^{\kappa+2} \tag{2.4.13}$$

and such that

$$R(u) \stackrel{def}{=} \frac{d}{dt}\big[\Theta_s(u(t,\cdot)) - \Theta_s^1(u(t,\cdot),\epsilon) - \Theta_s^2(u(t,\cdot)) - \Theta_s^3(u(t,\cdot)) - \Theta_s^4(u(t,\cdot))\big] \tag{2.4.14}$$

satisfies

$$|R(u)| \leq C_s \epsilon^{-(4-a+\rho)\theta\kappa} \|u\|_{\mathcal{H}^s}^{2\kappa+2} + C_s \epsilon^{(a-1)\theta\kappa} \|u\|_{\mathcal{H}^s}^{\kappa+2} + C_s \|u\|_{\mathcal{H}^s}^{2\kappa+2}. \tag{2.4.15}$$

Proof. Considering the right hand side of (2.2.6), we decompose

$$M_\ell^p(u_1, \ldots, u_{p+1}) = M_\ell^{p,\epsilon}(u_1, \ldots, u_{p+1}) + V_\ell^{p,\epsilon}(u_1, \ldots, u_{p+1}), \tag{2.4.16}$$

where the first term is given by (2.4.3) and the second one by

$$V_\ell^{p,\epsilon}(u_1, \ldots, u_{p+1}) = \sum_{n_0} \sum_{n_{p+1}} \mathbf{1}_{\{\sqrt{n_0} + \sqrt{n_{p+1}} \geq \epsilon^{-\theta\kappa}\}} \Pi_{n_0} M_\ell^p(u_1, \ldots, u_p, \Pi_{n_{p+1}} u_{p+1}). \tag{2.4.17}$$

By Definition 2.1.1, we get for $a = 2$ if $d \geq 2$ and $a = \frac{13}{6} - \varsigma$ if $d = 1$

$$\begin{aligned} &\|V_\ell^{p,\epsilon}(u_1, \ldots, u_{p+1})\|_{\mathcal{H}^{-s}} \\ &\quad \leq C_N \sum_{n_0} \cdots \sum_{n_{p+1}} (1 + \sqrt{n_0} + \sqrt{n_{p+1}})^{2s-a} \frac{(1+\sqrt{n'})^\nu \mu(n_0, \ldots, n_{p+1})^N}{S(n_0, \ldots, n_{p+1})^N} \\ &\qquad \times \mathbf{1}_{\{\sqrt{n_0} + \sqrt{n_{p+1}} \geq \epsilon^{-\theta\kappa}, |n_0 - n_{p+1}| < \frac{1}{2}(n_0 + n_{p+1}), n' \leq n_{p+1}\}} (1 + \sqrt{n_0})^{-s} \prod_{j=1}^{p+1} \|\Pi_{n_j} u_j\|_{L^2}. \end{aligned} \tag{2.4.18}$$

Following the proof of Proposition 2.1.2, we know that the gain of a powers of $\sqrt{n_0} + \sqrt{n_{p+1}}$ in the first term in the right hand side, coming from the fact that $M_\ell^p \in \mathcal{M}_{p+1}^{\nu, 2s-a}$, together with the condition $\sqrt{n_0} + \sqrt{n_{p+1}} \geq \epsilon^{-\theta\kappa}$, allows us to estimate, for N large enough and s_0 large enough with respect to ν, (2.4.18) by $C\epsilon^{(a-1)\theta\kappa} \Pi_{j=1}^p \|u_j\|_{\mathcal{H}^{s_0}} \|u_{p+1}\|_{\mathcal{H}^s}$. Consequently, the quantity

$$\sum_{p=\kappa}^{2\kappa-1} \sum_{\ell=0}^{p} Re\, i\langle V_\ell^{p,\epsilon}(\bar{u}, \ldots, \bar{u}, u, \ldots, u), u\rangle \tag{2.4.19}$$

is bounded form above by the second term of the right hand side of (2.4.15). In the rest of the proof, we may therefore replace in the right hand side of (2.2.6) M_ℓ^p by $M_\ell^{p,\epsilon}$.

Apply Lemma 2.4.1 to $M_\ell^{p,\epsilon}$, $\widetilde{M}_\ell^p$, R_ℓ^p, $\widetilde{R}_\ell^p$. This gives $\underline{M}_\ell^{p,\epsilon}$, $\underline{M}_\ell^p$, $\underline{R}_\ell^p$, $\underline{R}_\ell^{\prime p}$. We set

$$\Theta_s^1(u(t,\cdot),\epsilon) = \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\langle \underline{M}_\ell^{p,\epsilon}(\bar{u},\dots,\bar{u},u,\dots,u),u\rangle,$$

$$\Theta_s^2(u(t,\cdot)) = \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\langle \underline{M}_\ell^{p}(\bar{u},\dots,\bar{u},u,\dots,u,\bar{u}),u\rangle,$$

$$\Theta_s^3(u(t,\cdot)) = \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\langle \underline{R}_\ell^{p}(\bar{u},\dots,\bar{u},u,\dots,u),u\rangle,$$

$$\Theta_s^4(u(t,\cdot)) = \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\langle \underline{R}_\ell^{\prime p}(\bar{u},\dots,\bar{u},u,\dots,u,\bar{u}),u\rangle. \tag{2.4.20}$$

The general term in $\Theta_s^1(u(t,\cdot),\epsilon)$ has modulus bounded from above by

$$\|\underline{M}_\ell^{p,\epsilon}(\bar{u},\dots,\bar{u},u,\dots,u)\|_{\mathscr{H}^{-s}}\|u\|_{\mathscr{H}^s} \le C\epsilon^{-(4-a+\rho)\theta\kappa}\|u\|_{\mathscr{H}^s}^{\kappa}\|u\|_{\mathscr{H}^s}^2$$

for u in the unit ball of $\mathscr{H}^s(\mathbb{R}^d)$, using Proposition 2.1.2 with $\tau = 2s-1$ and Proposition 1.1.19 and (2.4.5). This gives the first inequality of (2.4.13). To obtain the other estimates in (2.4.13), we apply Proposition 2.1.2 to $\underline{M}_\ell^p$, remarking that if in (2.1.3) $\tau = 2s-1$ and s is large enough, the left hand side of (2.1.3) controls the $\mathscr{H}^{-s}$ norm of $\underline{M}_\ell^p(\bar{u},\dots,\bar{u},u,\dots,u,\bar{u})$. We also apply Proposition 2.1.5 with $\tau = 2s$ in (2.1.18) to $\underline{R}_\ell^p$, $\underline{R}_\ell^{\prime p}$. Then if s_0 is large enough, the left hand side of (2.1.18) controls $\mathscr{H}^{-s}$ norm of $\underline{R}_\ell^p(\bar{u},\dots,\bar{u},u,\dots,u)$ and $\underline{R}_\ell^{\prime p}(\bar{u},\dots,\bar{u},u,\dots,u,\bar{u})$. These give us the other inequalities in (2.4.13). Consequently we are left with proving (2.4.15). Remarking that we may also write the equation as

$$(D_t - \Lambda_m)u = -F\left(\Lambda_m^{-1}\left(\frac{u+\bar{u}}{2}\right)\right), \tag{2.4.21}$$

we compute using notation (2.4.1)

$$\begin{aligned}\frac{d}{dt}\Theta_s^1(u,\epsilon) = &\sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\, i\langle L_\ell^-(\underline{M}_\ell^{p,\epsilon})(\bar{u},\dots,\bar{u},u,\dots,u),u\rangle\\ &+\sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p}\sum_{j=1}^{\ell} Re\, i\langle \underline{M}_\ell^{p,\epsilon}(\bar{u},\dots,\overline{F},\dots,\bar{u},u,\dots,u),u\rangle\\ &-\sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p}\sum_{j=\ell+1}^{p+1} Re\, i\langle \underline{M}_\ell^{p,\epsilon}(\bar{u},\dots,\bar{u},u,\dots,F,\dots,u),u\rangle\\ &+\sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\, i\langle \underline{M}_\ell^{p,\epsilon}(\bar{u},\dots,\bar{u},u,\dots,u),F\rangle.\end{aligned} \tag{2.4.22}$$

By assumption on F, we have by Propositions 1.1.19 and 1.1.21 that $\|F(v)\|_{\mathscr{H}^s} \le C\|u\|_{\mathscr{H}^s}^{\kappa}\|u\|_{\mathscr{H}^s}$ if s is large enough and $\|u\|_{\mathscr{H}^s}\le 1$. Since $\underline{M}_\ell^{p,\epsilon} \in \widetilde{\mathscr{M}}_{p+1}^{\nu+\bar{\nu},2s-1}(\omega_\ell)$, we may apply Proposition 2.1.2 with $\tau = 2s-1$ and (2.4.5) to see that the last three terms in (2.4.22) have modulus bounded from above by the first term in the right hand side of (2.4.15). When computing $\frac{d}{dt}\Theta_s(u)$, noting that we have replaced M_ℓ^p by $M_\ell^{p,\epsilon}$, the first term in the right hand side of (2.2.6) is the first term in the right hand side of (2.4.22) because of (2.4.4). Consequently, these contributions will cancel out each other in the expression $\frac{d}{dt}[\Theta_s(u) - \Theta_s^1(u,\epsilon)]$. We compute

$$
\begin{aligned}
\frac{d}{dt}\Theta_s^2(u) = & \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\, i\langle L_\ell^+(\underline{M}_\ell^p)(\bar{u},\ldots,\bar{u},u,\ldots,u,\bar{u}),u\rangle \\
& + \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p}\sum_{j=1}^{\ell} Re\, i\langle \underline{M}_\ell^p(\bar{u},\ldots,\overline{F},\ldots,\bar{u},u,\ldots,u,\bar{u}),u\rangle \\
& - \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p}\sum_{j=\ell+1}^{p} Re\, i\langle \underline{M}_\ell^p(\bar{u},\ldots,\bar{u},u,\ldots,F,\ldots,u,\bar{u}),u\rangle \\
& + \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\, i\langle \underline{M}_\ell^p(\bar{u},\ldots,\bar{u},u,\ldots,u,\overline{F}),u\rangle \\
& + \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\, i\langle \underline{M}_\ell^p(\bar{u},\ldots,\bar{u},u,\ldots,u,\bar{u}),F\rangle. \qquad (2.4.23)
\end{aligned}
$$

Since $\underline{M}_\ell^p \in \widetilde{\mathscr{M}}_{p+1}^{\nu+\bar{\nu},2s-2}(\tilde{\omega}_\ell)$, we have by Proposition 2.1.2 with $\tau = 2s-1$, Proposition 1.1.19 and (2.4.5) that the last three terms are estimated by the last term in the right hand side of (2.4.15) if s is large enough. The first one, according to Lemma 2.4.1, cancels the contribution of $\widetilde{M}_\ell^p$ in (2.2.6) when computing $R(u)$. We may treat $\Theta_s^3(u)$ and $\Theta_s^4(u)$ in the same way using Proposition 2.1.5 with $\tau = 2s$, and this will lead to the third term in the right hand side of (2.4.15). Finally, the last term in (2.2.6) contributes to the last term in the right hand side of (2.4.15). This concludes the proof of the proposition. □

Proof of Theorem 2.1.1. We deduce from (2.4.13) and (2.4.15)

$$
\begin{aligned}
\Theta_s(u(t,\cdot)) \le\ & \Theta_s(u(0,\cdot)) - \Theta_s^1(u(0,\cdot),\epsilon) - \Theta_s^2(u(0,\cdot)) - \Theta_s^3(u(0,\cdot)) - \Theta_s^4(u(0,\cdot)) \\
& + \Theta_s^1(u(t,\cdot),\epsilon) + \Theta_s^2(u(t,\cdot)) + \Theta_s^3(u(t,\cdot)) + \Theta_s^4(u(t,\cdot)) \\
& + C_s\epsilon^{-(4-a+\rho)\theta\kappa}\int_0^t \|u(t',\cdot)\|_{\mathscr{H}^s}^{2\kappa}\|u(t',\cdot)\|_{\mathscr{H}^s}^2 dt' \\
& + C_s\epsilon^{(a-1)\theta\kappa}\int_0^t \|u(t',\cdot)\|_{\mathscr{H}^s}^{\kappa}\|u(t',\cdot)\|_{\mathscr{H}^s}^2 dt' \\
& + C_s\int_0^t \|u(t',\cdot)\|_{\mathscr{H}^s}^{2\kappa}\|u(t',\cdot)\|_{\mathscr{H}^s}^2 dt', \qquad (2.4.24)
\end{aligned}
$$

where $a = 2$ if $d \ge 2$ and $a = \frac{13}{6} - \varsigma$ for any $\varsigma \in (0,1)$ if $d = 1$. Take $\theta = \frac{1}{3+\rho}$ and $B > 1$ a constant such that for any (v_0, v_1) in the unit ball of $\mathscr{H}^{s+1}(\mathbb{R}^d)\times\mathscr{H}^s(\mathbb{R}^d)$, $u(0,\cdot) = \epsilon(-iv_1 + \Lambda_m v_0)$ satisfies $\|u(0,\cdot)\|_{\mathscr{H}^s} \le B\epsilon$. Let $K > B$ be another constant to be chosen, and assume that for τ' in some interval $[0,T]$ we have $\|u(\tau',\cdot)\|_{\mathscr{H}^s} \le K\epsilon \le 1$. If $d \ge 2$, using (2.4.13) with $a = 2$ we deduce from (2.4.24) and that there is a constant $C > 0$, independent of B, K, ϵ, such that as long as $t \in [0,T]$

$$
\|u(t,\cdot)\|_{\mathscr{H}^s}^2 \le C[B^2 + \epsilon^{\frac{1}{3+\rho}\kappa}K^{\kappa+2} + t\epsilon^{\frac{4+\rho}{3+\rho}\kappa}(K^{2\kappa+2}+K^{\kappa+2}) + t\epsilon^{2\kappa}K^{2\kappa+2}]\epsilon^2.
$$

If we assume that $T \leq c\epsilon^{-\frac{4+\rho}{3+\rho}\kappa}$, where $\rho > 0$ is arbitrary, for a small enough $c > 0$, and that ϵ is small enough, we get $\|u(t,\cdot)\|^2_{\mathscr{H}^s} \leq C(2B^2)\epsilon^2$. If K has been chosen initially so that $2CB^2 < K^2$, we get by a standard continuity argument that the priori bound $\|u(t,\cdot)\|_{\mathscr{H}^s} \leq K\epsilon$ holds true on $[0, c\epsilon^{-\frac{4+\rho}{3+\rho}\kappa}]$, in other words, the solution extends to such an interval $|t| \leq c\epsilon^{-\frac{4}{3}(1-\rho)\kappa}$ with another arbitrary $\rho > 0$. If $d = 1$, we may use (2.4.13) with $a = \frac{13}{6} - \varsigma$ to get

$$\|u(t,\cdot)\|^2_{\mathscr{H}^s} \leq C[B^2 + \epsilon^{\frac{7-6\varsigma}{18+6\rho}\kappa}K^{\kappa+2} + t\epsilon^{\frac{25+6(\rho-s)}{18+6\rho}\kappa}(K^{2\kappa+2} + K^{\kappa+2}) + t\epsilon^{2\kappa}K^{2\kappa+2}]\epsilon^2.$$

With the same reasoning we may get in this case that the solution extends to an interval of $|t| < c\epsilon^{-\frac{25}{18}(1-\rho)\kappa}$ for some small $c > 0$ and any $\rho > 0$. This concludes the proof of the theorem. □

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§1.2

Long-Time Existence for Semi-Linear Klein-Gordon Equations on Tori

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Long-time existence for semi-linear Klein–Gordon equations on tori ☆

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ABSTRACT

We use normal forms for Sobolev energy to prove that small smooth solutions of semi-linear Klein–Gordon equations on the torus exist over a larger interval than the one given by local existence theory, for almost every value mass. The gain on the length of the lifespan does not depend on the dimension. The result relies on the fact that the difference of square of two successive distinct eigenvalues of $\sqrt{-\Delta}$ on $\mathbb{T}^d$ can be bounded from below by a constant.

1. Introduction

Our main result concerns long-time existence for solutions to semi-linear Klein–Gordon equations on the torus of type

$$
\begin{aligned}
\left(\partial_t^2 - \Delta + m^2\right) v &= v^{\kappa+1}, \\
v|_{t=0} &= \epsilon v_0, \\
\partial_t v|_{t=0} &= \epsilon v_1
\end{aligned}
\tag{1.1}
$$

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where $(t,x) \in \mathbb{R}_+ \times \mathbb{T}^d$, $m \in \mathbb{R}_+^*$, $(v_0, v_1) \in H^{s+1}(\mathbb{T}^d) \times H^s(\mathbb{T}^d)$ for a large enough s and where $\epsilon > 0$ is small enough.

This problem has been studied in dimension 1 by Bourgain [4], Bambusi [1], Bambusi and Grébert [3]. They showed that one has almost global existence: for any N, if the data are in $H^{s+1} \times H^s$ for some s depending on N, if m stays outside an exceptional subset of zero measure, the solution exists at least on an interval of length $C_N \epsilon^{-N}$. For the problem in dimension at least 2, Delort and Szeftel [6] proved that the solution is defined on an interval of length at least $c\epsilon^{-2}$, if the nonlinearities vanish at the origin at order $\kappa + 1 = 2$. Recently, it was shown in Delort [5] that, for dimension $d \geqslant 2$ and for nonlinearities vanishing at the origin at order $\kappa + 1$ with any $\kappa \in \mathbb{N}^*$, the solutions extend at least over an interval of length $c\epsilon^{-\kappa(1+2/d)}$ up to a logarithm. Note that the gain of the power $\alpha = 2/d$, in comparison with the result given by local theory, depends on dimension d and becomes smaller and smaller as dimension d goes to infinity. A natural question is: Is it possible to obtain for such a Cauchy problem a solution defined on $c\epsilon^{-\kappa(1+\alpha)}$ with $\alpha > 0$ explicit and independent of dimension d? This paper gives a positive answer to this question. In fact, we prove that α can be taken to be a constant as close to $1/2$ as wanted. This is better than the result of Delort [5] when dimension d is larger than 4.

The method we use is based on normal form methods. Such an idea has been introduced in the study of nonlinear Klein–Gordon equations on $\mathbb{R}^d$ by Shatah [9] and is at the root of the results obtained on $\mathbb{S}^1$, $\mathbb{S}^d$, $\mathbb{T}^d$ in [4,1,3,2,5]. We also refer to Delort and Szeftel [6,7] for an application of this idea when one studies long-time existence of the same Cauchy problem on spheres and Zoll manifolds. And in Zhang [10], the author used such an idea to obtain a lower bound for the lifespan of solutions to $(\partial_t^2 - \Delta + |x|^2 + m^2)v = v^{\kappa+1}$ in $\mathbb{R}^d$ with small smooth Cauchy data. The proof there implies that the multiplicity of eigenvalues of $\sqrt{-\Delta + |x|^2}$ on $\mathbb{R}^d$ does not play any role and the gain on the exponent is independent of dimension d. Enlightened by that, we solve the problem we have just posed.

For the convenience of the reader, let us explain the idea more clearly with model (1.1) though it is similar to that of [5]. The goal is to control the Sobolev energy computing

$$\frac{d}{dt}\left[\left\|v(t,\cdot)\right\|_{H^{s+1}}^2 + \left\|\partial_t v(t,\cdot)\right\|_{H^s}^2\right]. \tag{1.2}$$

Using the equation, we may write this quantity as a multilinear expression in v, $\partial_t v$ homogeneous of degree $\kappa + 2$. We then perturb the Sobolev energy by an expression homogeneous of degree $\kappa + 2$ so that its time derivative cancel out the contribution in (1.2), up to reminders of higher order. The difficulty is to construct the perturbation in such a way that it can be controlled by powers of $\|v(t,\cdot)\|_{H^{s+1}} + \|\partial_t v(t,\cdot)\|_{H^s}$, with the same s in (1.2). Using expansion of elements of H^s on a basis of L^2 made of eigenfunctions of $\sqrt{-\Delta}$, we are reduced to study the expression of type

$$\sum_{n_0,\dots,n_{\kappa+1}} F_m(\lambda_{n_0},\dots,\lambda_{n_{\kappa+1}})^{-1} \int_{\mathbb{T}^d} (\Pi_{\lambda_{n_0}} u_0) \cdots (\Pi_{\lambda_{n_{\kappa+1}}} u_{\kappa+1})(\lambda_{n_0} + \cdots + \lambda_{n_{\kappa+1}})^{2s} \tag{1.3}$$

where λ_{n_j} are eigenvalues of $\sqrt{-\Delta}$ on $\mathbb{T}^d$, Π_λ is the spectral projector associated to the eigenvalue λ, and F_m is given by

$$F_m(\xi_0,\dots,\xi_{\kappa+1}) = \sum_{j=0}^{\kappa+1} e_j \sqrt{m^2 + \xi_j^2}, \quad e_j \in \{-1, 1\}. \tag{1.4}$$

The problem is to bound $|F_m(\lambda_{n_0},\dots,\lambda_{n_{\kappa+1}})|$ from below, for those λ_{n_j} for which (1.3) is nonzero, in such a way that (1.3) can be bounded from above by $C \prod \|u_j\|_{H^s}$ for s large enough. We assume for simplification that κ is odd and that λ_{n_0}, $\lambda_{n_{\kappa+1}}$ are the largest two among $\lambda_{n_0},\dots,\lambda_{n_{\kappa+1}}$. We divide it into two cases according to the estimate of F_m. The first case is $e_0 e_{\kappa+1} = 1$. We have (see Proposition 7.2), in this case, for almost all $m > 0$, there are $c > 0$ and $N_0 \in \mathbb{N}$ with

$$\begin{aligned}\left|F_m(\lambda_{n_0}, \ldots, \lambda_{n_{\kappa+1}})\right| \geqslant c(1 + \lambda_{n_0} + \lambda_{n_{\kappa+1}}) \\ \times \left(1 + \text{the third largest among}(\lambda_{n_0}, \ldots, \lambda_{n_{\kappa+1}})\right)^{-N_0}\end{aligned} \tag{1.5}$$

for any $n_0, \ldots, n_{\kappa+1} \in \mathbb{N}$. Plugging (1.5) into (1.3), we see that the loss is given by a large power of a small frequency, which allow us to estimate (1.3) by $C \prod_j \|u_j\|_{H^s}$ for $s \gg N_0$. This case corresponds to terms $\widetilde{M}_\ell^p$ in Section 6 when $|\lambda_{n_0} - \lambda_{n_{\kappa+1}}| \leqslant \frac{1}{2}(\lambda_{n_0} + \lambda_{n_{\kappa+1}})$, and to terms $\widetilde{R}_\ell^p$ when $|\lambda_{n_0} - \lambda_{n_{\kappa+1}}| > \frac{1}{2}(\lambda_{n_0} + \lambda_{n_{\kappa+1}})$. The second case is $e_0 e_{\kappa+1} = -1$. We shall show (see Proposition 7.1) for any $\rho > 0$, for almost $m > 0$, there are $c > 0$ and $N_0 \in \mathbb{N}$ such that

$$\left|F_m(\lambda_{n_0}, \ldots, \lambda_{n_{\kappa+1}})\right| \geqslant c(1 + \lambda_{n_0} + \lambda_{n_{\kappa+1}})^{-3-\rho} \times \left(|\lambda_{n_0} - \lambda_{n_{\kappa+1}}| + \lambda_{n_1} + \cdots + \lambda_{n_\kappa}\right)^{-N_0} \tag{1.6}$$

for any $n_0, \ldots, n_{\kappa+1} \in \mathbb{N}$. Note that this inequality is independent of the dimension d and better than the corresponding one of [5] when the dimension $d \geqslant 4$. This is the key point for us to improve the results of [5]. Plugging (1.6) into (1.3), we then see that when dividing by F_m there is not only a loss of a power of low frequencies which is harmless, but also a loss of $3 + \rho$ derivatives of high frequencies. However, solving the linear equation makes gain one derivative since the nonlinearity involves no derivative of v and we may gain one more derivative through commutators. This allows us to recover the loss and get an upper bound by $C \prod \|u_j\|_{H^s}$ of (1.3) through partition of frequencies between zones $\{\lambda_{n_j} \leqslant \epsilon^{-\kappa\theta},\ j = 1, \ldots, \kappa + 1\}$ and $\{\lambda_{n_j} > \epsilon^{-\kappa\theta}$ for at least one $j \in \{1, \ldots, \kappa + 1\}\}$, where θ is a constant to be chosen.

In comparison with the method of [5], we have to overcome several difficulties in the above process. The first one is to find out a way so that one can get a dimension-independent estimate of small divisors, that is (1.6). In fact, we can use the projectors on the eigenspaces associated to different eigenvalues of $\sqrt{-\Delta}$ on $\mathbb{T}^d$, instead of the projectors on the space spanned by each eigenfunction which were exclusively used in [5]. From this point of view, the multiplicity of the eigenvalues does not play any role while it does in [5]. This implies that the estimates we want may be independent of the dimension. However, when one tries to extend multilinear operators to Sobolev spaces, a loss of one derivative is inevitable in this framework (see Proposition 5.2) because of the bad behavior of the eigenvalues of $\sqrt{-\Delta}$ on $\mathbb{T}^d$. So another difficulty is to find a technique to avoid such a loss for the high frequency part of the nonlinearity to which we shall not use normal forms. But above all, one has to prove (1.6) which is independent of the dimension. This can be done by noting that the eigenvalues of the harmonic oscillator $\sqrt{-\Delta + |x|^2}$ on $\mathbb{R}^d$ and those of $\sqrt{-\Delta}$ on $\mathbb{T}^d$ share similar properties and we have already had an estimate of that type in the case of the harmonic oscillator. The point is that when the dimension increases, the multiplicity of the eigenvalues of $\sqrt{-\Delta}$ on $\mathbb{T}^d$ grows, while the spacing between different eigenvalues remains essentially the same.

We state our main result in Section 2 and after introducing some notations we obtain some properties of eigenvalues and spectral projections in Section 4. Then we define some multilinear operator spaces so that we may rewrite Sobolev energy in terms of elements in these spaces. This is done in Section 5 and 6. The last two sections are devoted to prove boundedness of Sobolev energy, which implies the main theorem.

2. Statement of the main theorem

Let $d \geqslant 2$ and set $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ for the standard torus. Denote by $\Box = \partial_t^2 - \Delta$ the D'Alembertian on $\mathbb{R} \times \mathbb{T}^d$. Let $F : \mathbb{T}^d \to \mathbb{R}$, $v \to F(v)$ be a real valued smooth function. We shall assume

$$\partial_v^j F(0) = 0 \quad \text{for } j = 0, \ldots, \kappa \tag{2.1}$$

for some $\kappa \in \mathbb{N}^*$. Let $m \in \mathbb{R}_+^*$. We consider the solution v of the Klein–Gordon equation

$$\begin{aligned}\left(\Box + m^2\right)v &= F(v),\\ v|_{t=0} &= \epsilon v_0,\\ \partial_t v|_{t=0} &= \epsilon v_1\end{aligned} \tag{2.2}$$

where $v_0 \in H^{s+1}(\mathbb{T}^d, \mathbb{R})$, $v_1 \in H^s(\mathbb{T}^d, \mathbb{R})$, and $\epsilon > 0$ is small enough. From Delort [5], we know that if s is large enough and $\epsilon \in (0, 1)$ small enough, Eq. (2.2) admits for any (v_0, v_1) in the unit ball of $H^{s+1} \times H^s$ a unique smooth solution defined on the interval $(-T_\epsilon, T_\epsilon)$ with $T_\epsilon \geqslant c\epsilon^{-\kappa(1+2/d)}|\log\epsilon|^{-A}$ for some uniform constant $c > 0$ and any $A > 1$. Moreover, $\|v(t,\cdot)\|_{H^{s+1}} + \|\partial_t v(t,\cdot)\|_{H^s}$ may be controlled by $C\epsilon$, for another uniform constant $C > 0$, on the interval of existence. The goal of this paper is to show that under convenient assumptions, we may obtain a solution on an interval of length larger than $c\epsilon^{-\kappa(1+1/2-\rho)}$ for some constant $c > 0$ and for any $\rho > 0$. Let us state the main result.

Theorem 2.1. *For any $\rho > 0$, there is a zero measure subset $\mathcal{N}$ of $\mathbb{R}_+^*$ such that for every $m \in \mathbb{R}_+^* - \mathcal{N}$, there are $\epsilon_0 > 0$, $c > 0$, $s_0 > 0$ such that for any $s \geqslant s_0$, any $\epsilon \in (0, \epsilon_0)$, any pair (v_0, v_1) of real valued functions belonging to the unit ball of $H^{s+1}(\mathbb{T}^d) \times H^s(\mathbb{T}^d)$, problem* (2.2) *has a unique solution*

$$v \in C^0\left((-T_\epsilon, T_\epsilon), H^{s+1}\left(\mathbb{T}^d\right)\right) \cap C^1\left((-T_\epsilon, T_\epsilon), H^s\left(\mathbb{T}^d\right)\right)$$

with $T_\epsilon \geqslant c\epsilon^{-\kappa(3/2-\rho)}$. Moreover, the solution v is uniformly bounded in $H^{s+1}(\mathbb{T}^d)$ for $|t| \leqslant c\epsilon^{-\kappa(3/2-\rho)}$ and $\partial_t v$ is uniformly bounded in $H^s(\mathbb{T}^d)$ on the same interval.

3. Notations

For $k \in \mathbb{Z}^d$ we set

$$\varphi_k = \frac{1}{(2\pi)^{d/2}} e^{ikx} \tag{3.1}$$

so that $(\varphi_k)_{k\in\mathbb{Z}^d}$ is a Hilbertian basis of $L^2(\mathbb{T}^d, \mathbb{C})$. Let $\widetilde{\Pi}_k$ be the orthogonal projection on the span of φ_k. We have for $u \in L^2(\mathbb{T}^d, \mathbb{C})$

$$\widetilde{\Pi}_k u = \langle u, \varphi_k\rangle\varphi_k. \tag{3.2}$$

Denote by S the spectrum of $\sqrt{-\Delta}$ on $\mathbb{T}^d$, that is,

$$S = \left\{|k|;\ k \in \mathbb{Z}^d\right\}. \tag{3.3}$$

Let $(\lambda_n)_{n\in\mathbb{N}}$ be the sequence consisting of distinct elements of S defined by induction as follows:

$$\lambda_{n+1} = \inf\left\{|k|;\ k \in \mathbb{Z}^d \text{ and } |k| > \lambda_n\right\}; \qquad \lambda_0 = 0. \tag{3.4}$$

For $n \in \mathbb{N}$, we also denote by Π_n the orthogonal projection on the eigenspace associated to λ_n, that is, for $u \in L^2(\mathbb{T}^d, \mathbb{C})$

$$\Pi_n u = \sum_{k\in\mathbb{Z}^d,\ |k|=\lambda_n} \langle u, \varphi_k\rangle\varphi_k. \tag{3.5}$$

For $n_0, \dots, n_{p+1}$ $p+2$ natural numbers, we shall also use the following notions throughout the paper

$$\vec{n} = (n_0, \dots, n_{p+1}) \in \mathbb{N}^{p+2},$$
$$\tilde{n} = (n_1, \dots, n_p) \in \mathbb{N}^p,$$
$$\lambda_{n'} = \lambda_{n_1} + \dots + \lambda_{n_p}, \tag{3.6}$$

where λ_{n_j}'s are defined in (3.4).

Finally, let $\mathcal{E}$ denote the space of trigonometric polynomials.

4. Properties of eigenvalues and spectral projectors on $\mathbb{T}^d$

For the sequence of eigenvalues defined in (3.4), we have the following properties.

Lemma 4.1. *Let* $(\lambda_n)_{n\in\mathbb{N}}$ *be defined by* (3.4) *and* S *by* (3.3). *Then*:

(i) $(\lambda_n)_{n\in\mathbb{N}}$ *is a nonnegative strictly increasing sequence*;
(ii) $|\lambda_{n+1}^2 - \lambda_n^2| \geqslant 1$, *for any* $n \in \mathbb{N}$;
(iii) $|\lambda_{n+1} - \lambda_n| \geqslant 1/(2\lambda_{n+1})$ *for any* $n \in \mathbb{N}$;
(iv) *Let* $N > 2$ *and* $A \geqslant 1$. *Then there exists* $C > 0$ *such that for any* $\ell \in \mathbb{N}$

$$\sum_{n\in\mathbb{N}} \big(|\lambda_n - \lambda_\ell| + A\big)^{-N} \leqslant C\big(A^{-(N-1)}\lambda_\ell + A^{-(N-2)}\big). \tag{4.1}$$

In particular, we have

$$\sum_{n\in\mathbb{N}} (\lambda_n + A)^{-N} \leqslant C A^{-(N-2)}. \tag{4.2}$$

Proof. (i) is an immediate result of the construction of $(\lambda_n)_n$. From the definition we know that there are $k \in \mathbb{Z}^d$, $\tilde{k} \in \mathbb{Z}^d$ such that $\lambda_{n+1} = |k|$, $\lambda_n = |\tilde{k}|$. Since $\lambda_{n+1} > \lambda_n$, $k \neq \tilde{k}$. Therefore, (ii) holds true. From (i) and (ii) we know that (iii) holds true. We are left to prove (iv). Compute

$$\begin{aligned}\sum_{n\in\mathbb{N}} \big(|\lambda_n - \lambda_\ell| + A\big)^{-N} &= A^{-N} + \sum_{n\in\mathbb{N},\, n\neq\ell} \sum_{j\in\mathbb{Z}} \big(|\lambda_n - \lambda_\ell| + A\big)^{-N} \mathbf{1}_{\{2^{j-1} \leqslant |\lambda_n - \lambda_\ell| < 2^j\}} \\ &\leqslant A^{-N} + C\sum_{j\in\mathbb{Z}} \big(2^j + A\big)^{-N} \sum_{n\in\mathbb{N},\, n\neq\ell} \mathbf{1}_{\{2^{j-1} \leqslant |\lambda_n - \lambda_\ell| < 2^j\}}. \end{aligned} \tag{4.3}$$

Let us estimate the number of λ_n's satisfying $2^{j-1} \leqslant |\lambda_n - \lambda_\ell| < 2^j$. For such λ_n's, $\lambda_n \in (\lambda_\ell - 2^j, \lambda_\ell + 2^j)$. From (iii) we know that the distance between two successive λ_n's staying in such an interval may be bounded from below by $c/(\lambda_\ell + 2^j)$ for some $c > 0$. Consequently, the number of such λ_n's is not bigger than $C2^j\lambda_\ell$ if $\lambda_\ell > \widetilde{C}2^j$, and than $C2^{2j}$ if $\lambda_\ell \leqslant \widetilde{C}2^j$. Therefore by (4.3) we have

$$\sum_{n\in\mathbb{N}} \big(|\lambda_n - \lambda_\ell| + A\big)^{-N} \leqslant A^{-N} + C\sum_{j\in\mathbb{Z}} \big(2^j\lambda_\ell + 2^{2j}\big)\big(2^j + A\big)^{-N}$$

$$\leqslant A^{-N} + C(\lambda_\ell + C_1)A^{-N} + C\sum_{j\in\mathbb{N}}\big(2^j\lambda_\ell + 2^{2j}\big)\big(2^j + A\big)^{-N}$$
$$\leqslant C\big(A^{-(N-1)}\lambda_\ell + A^{-(N-2)}\big). \tag{4.4}$$

This concludes the proof. □

Lemma 4.2. *Let $v \in H^s(\mathbb{T}^d)$, $s \geqslant [d/2] + 1$ and assume that $F \in C^\infty(\mathbb{R})$ vanishes at the origin at order p, $p \in \mathbb{N}^*$. Then we have $F(v) \in H^s(\mathbb{T}^d)$. Moreover, $\|F(v)\|_{H^s} \leqslant C_s\|v\|^p_{H^s}$ for some $C_s > 0$.*

Proof. Since $\mathbb{T}^d$ is a compact manifold, using the finite cover theorem and partitions of unity, we may reduce ourselves to working in local coordinate. Now the lemma follows from Corollary 6.4.5 in [8] and Sobolev inequality. □

We shall need the following lemmas.

Lemma 4.3. *For any $\gamma > d$, there is a constant C such that for any $A > 1$*

$$\sum_{k\in\mathbb{Z}^d}\big(|k| + A\big)^{-\gamma} \leqslant CA^{-(\gamma-d)}. \tag{4.5}$$

Proof. This follows from the facts that for any $N > 1$, $\sum_{n\in\mathbb{Z}}(|n| + A)^{-N} \leqslant CA^{-(N-1)}$ and that $\sum_{k\in\mathbb{Z}^d}(|k| + A)^{-\gamma} \leqslant C\sum_{k_1,\dots,k_d\in\mathbb{Z}}(|k_1| + \cdots + |k_d| + A)^{-\gamma}$. □

By definition, we have

Lemma 4.4. *For any $n \in \mathbb{N}$, any $k \in \mathbb{Z}^d$,*

$$\|\Pi_n\|_{\mathcal{L}(L^2,L^2)} \leqslant 1, \qquad \|\widetilde{\Pi}_k\|_{\mathcal{L}(L^2,L^2)} \leqslant 1. \tag{4.6}$$

It is known that the product of two eigenfunctions of $\sqrt{-\Delta}$ on the torus is another eigenfunction. This fact together with Lemma 4.4 and Sobolev embedding theorem gives

Lemma 4.5. *Let $p \in \mathbb{N}^*$. For any $k_0, k_{p+1} \in \mathbb{Z}^d$, any $n_0, \dots, n_{p+1} \in \mathbb{N}$, any $u_1, \dots, u_{p+1} \in L^2(\mathbb{T}^d, \mathbb{C})$,*

(i) *if $|k_0 - k_{p+1}| > \lambda_{n'}$ with $\lambda_{n'}$ defined by* (3.6)*, then*

$$\widetilde{\Pi}_{k_0}\big[(\Pi_{n_1}u_1)\cdots(\Pi_{n_p}u_p)(\widetilde{\Pi}_{k_{p+1}}u_{p+1})\big] \equiv 0;$$

(ii) *if $|\lambda_{n_0} - \lambda_{n_{p+1}}| > \lambda_{n'}$, then*

$$\Pi_{n_0}\big[(\Pi_{n_1}u_1)\cdots(\Pi_{n_p}u_p)(\Pi_{n_{p+1}}u_{p+1})\big] \equiv 0;$$

(iii) *one has for any $\nu > d/2$,*

$$\big\|\widetilde{\Pi}_{k_0}\big[(\Pi_{n_1}u_1)\cdots(\Pi_{n_p}u_p)(\widetilde{\Pi}_{k_{p+1}}u_{p+1})\big]\big\|_{L^2} \leqslant (1 + \lambda_{n'})^\nu\prod_{j=1}^{p+1}\|u_j\|_{L^2}, \tag{4.7}$$

$$\big\|\Pi_{n_0}\big[(\Pi_{n_1}u_1)\cdots(\Pi_{n_p}u_p)(\Pi_{n_{p+1}}u_{p+1})\big]\big\|_{L^2} \leqslant (1 + \lambda_{n'})^\nu\prod_{j=1}^{p+1}\|u_j\|_{L^2}. \tag{4.8}$$

5. Definitions and properties of multilinear operators

In this section we give some definitions of various spaces of multilinear operators we shall use. We also deduce some properties of operators in such spaces. Keep notations $\mathcal{E}$, $(\lambda_n)_{n\in\mathbb{N}}$ and $\lambda_{n'}$ defined in Section 3 in mind.

Definition 5.1. Let $\nu \in \mathbb{R}_+$, $\tau \in \mathbb{R}$, $\delta \in (0,1)$, $p \in \mathbb{N}^*$. We denote by $\mathcal{M}^{\nu,\tau}_{p+1,\delta}$ the space of all $p+1$-linear operators $(u_1,\dots,u_{p+1}) \to M(u_1,\dots,u_{p+1})$, defined on $\mathcal{E}\times\cdots\times\mathcal{E}$ with values in $L^2(\mathbb{T}^d)$ such that:

(i) For any $u_1,\dots,u_{p+1} \in \mathcal{E}$, one has

$$\Pi_{n_0}\big[M(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})\big] \equiv 0 \tag{5.1}$$

if $(n_0,\dots,n_{p+1}) \in \mathbb{N}^{p+2}$ satisfies one of the following conditions:
(a) $|\lambda_{n_0}-\lambda_{n_{p+1}}| \geqslant 1/2(\lambda_{n_0}+\lambda_{n_{p+1}})$;
(b) $|\lambda_{n_0}-\lambda_{n_{p+1}}| > \lambda_{n'}$;
(c) $\lambda_{n'} \geqslant \delta\lambda_{n_{p+1}}$.

(ii) There exists $C>0$ such that for any $(n_0,\dots,n_{p+1}) \in \mathbb{N}^{p+2}$, any $u_1,\dots,u_{p+1} \in \mathcal{E}$, one has

$$\big\|\Pi_{n_0}\big[M(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})\big]\big\|_{L^2} \leqslant C(1+\lambda_{n_0}+\lambda_{n_{p+1}})^\tau(1+\lambda_{n'})^\nu \prod_{j=1}^{p+1}\|u_j\|_{L^2}. \tag{5.2}$$

The best constant in the preceding inequality will be denoted by $\|M\|_{\mathcal{M}^{\nu,\tau}_{p+1,\delta}}$.

We may extend the operators in $\mathcal{M}^{\nu,\tau}_{p+1,\delta}$ to Sobolev spaces.

Proposition 5.2. *Let $\nu \in \mathbb{R}_+$, $\tau \in \mathbb{R}$, $\delta \in (0,1)$, $p \in \mathbb{N}^*$, $s > \nu+2$. Then any element $M \in \mathcal{M}^{\nu,\tau}_{p+1,\delta}$ extends as a bounded operator from $H^s(\mathbb{T}^d)\times\cdots\times H^s(\mathbb{T}^d)$ to $H^{s-\tau-1}(\mathbb{T}^d)$. Moreover, for any $s_0 \in (\nu+2,s]$, there is $C>0$ such that for any $M \in \mathcal{M}^{\nu,\tau}_{p+1,\delta}$ and any $u_1,\dots,u_{p+1} \in H^s(\mathbb{T}^d)$,*

$$\big\|M(u_1,\dots,u_{p+1})\big\|_{H^{s-\tau-1}} \leqslant C\|M\|_{\mathcal{M}^{\nu,\tau}_{p+1,\delta}} \prod_{j=1}^{p}\|u_j\|_{H^{s_0}}\|u_{p+1}\|_{H^s}. \tag{5.3}$$

Proof. We write

$$\begin{aligned}
&\big\|M(u_1,\dots,u_{p+1})\big\|^2_{H^{s-\tau-1}} \\
&\quad = \sum_{n_0}\bigg\|\sum_{n_1}\cdots\sum_{n_{p+1}}\Pi_{n_0}M(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})\bigg\|^2_{L^2}\big(1+\lambda^2_{n_0}\big)^{s-\tau-1}.
\end{aligned} \tag{5.4}$$

Because of (i) of Definition 5.1, using the symmetries, we may assume $(n_0,\dots,n_{p+1}) \in \mathbb{N}^{p+2}$ is such that

$$\begin{gathered}
|\lambda_{n_0}-\lambda_{n_{p+1}}| < 1/2(\lambda_{n_0}+\lambda_{n_{p+1}}), \\
\lambda_{n_1} \leqslant \cdots \leqslant \lambda_{n_p} < \delta\lambda_{n_{p+1}}, \\
|\lambda_{n_0}-\lambda_{n_{p+1}}| \leqslant \lambda_{n'}
\end{gathered} \tag{5.5}$$

when estimating the general term in the right-hand side of (5.4). From (5.5) we deduce

$$\lambda_{n_0} \sim \lambda_{n_{p+1}}, \qquad \lambda_{n'} \sim \lambda_{n_p}.$$

Therefore, we see from (5.2) that the square root of the general term in the n_0 sum in (5.4) is smaller than a constant times

$$\sum_{n_1 \leqslant \cdots \leqslant n_{p+1}} (1+\lambda_{n_{p+1}})^{s-1} (1+\lambda_{n_p})^{\nu} \bar{\mathbf{1}}_{\{n_0,\ldots,n_{p+1}\}} \prod_{j=1}^{p+1} \|\Pi_{n_j} u_j\|_{L^2}, \tag{5.6}$$

where

$$\bar{\mathbf{1}}_{\{n_0,\ldots,n_{p+1}\}} = \mathbf{1}_{\{|\lambda_{n_0}-\lambda_{n_{p+1}}| \leqslant C\lambda_{n_p},\ \lambda_{n_0}\sim\lambda_{n_{p+1}},\ \lambda_{n_p} \leqslant \delta\lambda_{n_{p+1}}\}}.$$

Now Hölder inequality allows us to bound (5.6) from above by a constant times the product of I and II, where I and II stand respectively for

$$\mathrm{I} = \left(\sum_{n_1 \leqslant \cdots \leqslant n_{p+1}} (1+\lambda_{n_p})^{\nu} \bar{\mathbf{1}}_{\{n_0,\ldots,n_{p+1}\}} \prod_{j=1}^{p} \|\Pi_{n_j} u_j\|_{L^2} \right)^{1/2},$$

$$\mathrm{II} = \left(\sum_{n_1 \leqslant \cdots \leqslant n_{p+1}} (1+\lambda_{n_{p+1}})^{2s-2} (1+\lambda_{n_p})^{\nu} \bar{\mathbf{1}}_{\{n_0,\ldots,n_{p+1}\}} \prod_{j=1}^{p} \|\Pi_{n_j} u_j\|_{L^2} \|\Pi_{n_{p+1}} u_{p+1}\|_{L^2}^2 \right)^{1/2}.$$

By (iv) of Lemma 4.1 we have

$$\sum_{n_0} \bar{\mathbf{1}}_{\{n_0,\ldots,n_{p+1}\}} \leqslant C \sum_{n_0} \frac{(1+\lambda_{n_p})^3 \mathbf{1}_{\{\lambda_{n_p} \leqslant \delta\lambda_{n_{p+1}}\}}}{(|\lambda_{n_0}-\lambda_{n_{p+1}}|+1+\lambda_{n_p})^3} \leqslant C(1+\lambda_{n_{p+1}})(1+\lambda_{n_p}),$$

$$\sum_{n_{p+1}} \bar{\mathbf{1}}_{\{n_0,\ldots,n_{p+1}\}} \leqslant C \sum_{n_{p+1}} \frac{(1+\lambda_{n_p})^3 \mathbf{1}_{\{\lambda_{n_p} \leqslant C\lambda_{n_0}\}}}{(|\lambda_{n_0}-\lambda_{n_{p+1}}|+1+\lambda_{n_p})^3} \leqslant C(1+\lambda_{n_0})(1+\lambda_{n_p}). \tag{5.7}$$

Thus using (5.7) to deal with n_{p+1} sum we get

$$\mathrm{I} \leqslant C \left[\sum_{n_1 \leqslant \cdots \leqslant n_p} (1+\lambda_{n_p})^{\nu+1} (1+\lambda_{n_0}) \prod_{j=1}^{p} \|\Pi_{n_j} u_j\|_{L^2} \right]^{1/2}$$

$$\leqslant C(1+\lambda_{n_0})^{1/2} \prod_{j=1}^{p} \|u_j\|_{H^{s_0}}^{1/2}$$

if we take $s_0 > \nu + 2$ using (4.2). We incorporate the factor $(1+\lambda_{n_0})^{1/2}$ coming from the term I into II and then compute

$$(1+\lambda_{n_0})^{1/2}\mathrm{II} \leqslant C\Bigg[\sum_{n_1\leqslant\cdots\leqslant n_{p+1}}(1+\lambda_{n_p})^{\nu}\bar{\mathbf{1}}_{\{|n_0,\ldots,n_{p+1}\}} \times (1+\lambda_{n_{p+1}})^{2s-1}\prod_{j=1}^{p}\|\Pi_{n_j}u_j\|_{L^2}\|\Pi_{n_{p+1}}u_{p+1}\|_{L^2}^2\Bigg]^{1/2}.$$

By the above analysis, we get

$$\begin{aligned}
\big\|M(u_1,\ldots,u_{p+1})\big\|_{H^{s-\tau-1}}^2 &\leqslant C\sum_{(n_0,\ldots,n_{p+1})\in\mathbb{N}^{p+2}}(1+\lambda_{n_p})^{\nu}\bar{\mathbf{1}}_{\{n_0,\ldots,n_{p+1}\}}(1+\lambda_{n_{p+1}})^{2s-1} \\
&\quad\times\prod_{j=1}^{p}\|\Pi_{n_j}u_j\|_{L^2}\|\Pi_{n_{p+1}}u_{p+1}\|_{L^2}^2\prod_{j=1}^{p}\|u_j\|_{H^{s_0}} \\
&\leqslant C\sum_{(n_1,\ldots,n_{p+1})\in\mathbb{N}^{p+1}}(1+\lambda_{n_p})^{\nu+1}(1+\lambda_{n_{p+1}})(1+\lambda_{n_{p+1}})^{2s-1} \\
&\quad\times\prod_{j=1}^{p}\|\Pi_{n_j}u_j\|_{L^2}\|\Pi_{n_{p+1}}u_{p+1}\|_{L^2}^2\prod_{j=1}^{p}\|u_j\|_{H^{s_0}} \\
&\leqslant C\prod_{j=1}^{p}\|u_j\|_{H^{s_0}}^2\|u_{p+1}\|_{H^s}^2,
\end{aligned} \tag{5.8}$$

where we have used (5.7) in the second inequality to handle n_0 sum and we have also taken $s>\nu+2$. The constant $\|M\|_{\mathcal{M}_{p+1,\delta}^{\nu,\tau}}$ is implicit in the constant C when we get the inequality (5.6). This concludes the proof. □

Let us define convenient subspaces of the spaces of Definition 5.1.

Definition 5.3. Let $\nu\in\mathbb{R}_+$, $\tau\in\mathbb{R}$, $\delta\in(0,1)$, $p\in\mathbb{N}^*$, $\omega:\{0,\ldots,p+1\}\to\{-1,1\}$ be given.

- If $\sum_{j=0}^{p+1}\omega(j)\neq 0$, we set $\widetilde{\mathcal{M}}_{p+1,\delta}^{\nu,\tau}(\omega)=\mathcal{M}_{p+1,\delta}^{\nu,\tau}$;
- If $\sum_{j=0}^{p+1}\omega(j)=0$, we denote by $\widetilde{\mathcal{M}}_{p+1,\delta}^{\nu,\tau}(\omega)$ the closed subspace of $\mathcal{M}_{p+1,\delta}^{\nu,\tau}$ given by those $M\in\mathcal{M}_{p+1,\delta}^{\nu,\tau}$ such that

$$\Pi_{n_0}M(\Pi_{n_1}u_1,\ldots,\Pi_{n_{p+1}}u_{p+1})\equiv 0 \tag{5.9}$$

for any $(n_0,\ldots,n_{p+1})\in S_p^{\omega}$, where

$$\begin{aligned}
S_p^{\omega}=\big\{&(n_0,\ldots,n_{p+1})\in\mathbb{N}^{p+2};\ \text{there is a bijection:}\\
&\sigma:\big\{0\leqslant j\leqslant p+1,\ \omega(j)=-1\big\}\to\big\{j;\ 0\leqslant j\leqslant p+1,\ \omega(j)=1\big\}\\
&\text{with } n_{\sigma(j)}=n_j \text{ for any } j \text{ in the first set}\big\}.
\end{aligned} \tag{5.10}$$

We shall need another subspace whose elements have better properties than those in $\mathcal{M}_{p+1,\delta}^{\nu,\tau}$.

Definition 5.4. Let $\nu \in \mathbb{R}_+$, $\tau \in \mathbb{R}$, $\delta \in (0,1)$, $p \in \mathbb{N}^*$. We denote by $\mathcal{G}^{\nu,\tau}_{p+1,\delta}$ the space of all $p+1$-linear operators $(u_1,\dots,u_{p+1}) \to M(u_1,\dots,u_{p+1})$, defined on $\mathcal{E} \times \cdots \times \mathcal{E}$ with values in $L^2(\mathbb{T}^d)$ such that:

(i) For any $u_1,\dots,u_{p+1} \in \mathcal{E}$, one has

$$\widetilde{\Pi}_{k_0}\big[M(\Pi_{n_1}u_1,\dots,\Pi_{n_p}u_p,\widetilde{\Pi}_{k_{p+1}}u_{p+1})\big] \equiv 0 \tag{5.11}$$

if $k_0, k_{p+1} \in \mathbb{Z}^d$, $(n_1,\dots,n_p) \in \mathbb{N}^p$ satisfy one of the following conditions:
(a) $|k_0 - k_{p+1}| \geqslant 1/2(|k_0| + |k_{p+1}|)$;
(b) $|k_0 - k_{p+1}| > \lambda_{n'}$;
(c) $\lambda_{n'} \geqslant \delta|k_{p+1}|$.

(ii) There is $C > 0$ such that for any $(n_1,\dots,n_p) \in \mathbb{N}^{p+2}$, any $k_0, k_{p+1} \in \mathbb{Z}^d$, any $u_1,\dots,u_{p+1} \in \mathcal{E}$, one has

$$\begin{aligned}
&\big\|\widetilde{\Pi}_{k_0}\big[M(\Pi_{n_1}u_1,\dots,\Pi_{n_p}u_p,\widetilde{\Pi}_{k_{p+1}}u_{p+1})\big]\big\|_{L^2} \\
&\quad \leqslant C\big(1 + |k_0| + |k_{p+1}|\big)^{\tau}(1+\lambda_{n'})^{\nu}\prod_{j=1}^{p+1}\|u_j\|_{L^2}.
\end{aligned} \tag{5.12}$$

The best constant in the preceding inequality will be denoted by $\|M\|_{\mathcal{G}^{\nu,\tau}_{p+1,\delta}}$.

Let us show that the space defined in Definition 5.4 is a subspace of that of Definition 5.1.

Proposition 5.5. *Let $\nu \in \mathbb{R}_+$, $\tau \in \mathbb{R}$, $\delta \in (0,1)$, $p \in \mathbb{N}^*$. Then we have*

$$\mathcal{G}^{\nu,\tau}_{p+1,\delta} \subset \mathcal{M}^{\nu+d,\tau}_{p+1,\delta}.$$

Proof. Let $M \in \mathcal{G}^{\nu,\tau}_{p+1,\delta}$. For any $(n_0,\dots,n_{p+1}) \in \mathbb{N}^{p+2}$ and any $u_1,\dots,u_{p+1} \in \mathcal{E}$,

$$\begin{aligned}
&\Pi_{n_0}M(\Pi_{n_1}u_1,\dots,\Pi_{n_p}u_p,\Pi_{n_{p+1}}u_{p+1}) \\
&\quad = \sum_{\substack{k_0,k_{p+1}\in\mathbb{Z}^d \\ |k_0|=\lambda_{n_0},\,|k_{p+1}|=\lambda_{n_{p+1}}}} \widetilde{\Pi}_{k_0}M(\Pi_{n_1}u_1,\dots,\Pi_{n_p}u_p,\widetilde{\Pi}_{k_{p+1}}u_{p+1}).
\end{aligned} \tag{5.13}$$

For the indices, which appear in (5.13), the conditions listed in (i) of Definition 5.1 imply the corresponding ones in (i) of Definition 5.4. Thus item (i) of Definition 5.1 is satisfied for M. Now let us establish (5.2). According to (5.13) the square of the quantity in the left-hand side of (5.2) equals

$$\sum_{\substack{k_0\in\mathbb{Z}^d \\ |k_0|=\lambda_{n_0}}} \bigg\| \sum_{\substack{k_{p+1}\in\mathbb{Z}^d \\ |k_{p+1}|=\lambda_{n_{p+1}}}} \widetilde{\Pi}_{k_0}M(\Pi_{n_1}u_1,\dots,\Pi_{n_p}u_p,\widetilde{\Pi}_{k_{p+1}}u_{p+1})\bigg\|^2_{L^2}. \tag{5.14}$$

The square root of the general term over k_0 sum in (5.14) is not larger than

$$\sum_{\substack{k_{p+1}\in\mathbb{Z}^d \\ |k_{p+1}|=\lambda_{n_{p+1}}}} \big\|\widetilde{\Pi}_{k_0}M(\Pi_{n_1}u_1,\dots,\Pi_{n_p}u_p,\widetilde{\Pi}_{k_{p+1}}u_{p+1})\big\|_{L^2}, \tag{5.15}$$

which, according to (5.12) and $|k_0| = \lambda_{n_0}$, may be bounded from above by a constant times

$$\sum_{\substack{k_{p+1}\in\mathbb{Z}^d\\ |k_{p+1}|=\lambda_{n_{p+1}}}} (1+\lambda_{n_0}+\lambda_{n_{p+1}})^{\tau}(1+\lambda_{n'})^{\nu}\mathbf{1}_{\{|k_0-k_{p+1}|\leqslant\lambda_{n'}\}}\prod_{j=1}^{p}\|u_j\|_{L^2}\|\widetilde{\Pi}_{k_{p+1}}u_{p+1}\|_{L^2}. \tag{5.16}$$

Therefore, applying Hölder inequality to the sum over k_{p+1} in (5.16), we get an upper bound of (5.16) by a constant times

$$\begin{aligned}&(1+\lambda_{n_0}+\lambda_{n_{p+1}})^{\tau}(1+\lambda_{n'})^{\nu}\prod_{j=1}^{p}\|u_j\|_{L^2}\\ &\quad\times\Bigg(\sum_{\substack{k_{p+1}\in\mathbb{Z}^d\\ |k_{p+1}|=\lambda_{n_{p+1}}}}\mathbf{1}_{\{|k_0-k_{p+1}|\leqslant\lambda_{n'}\}}\Bigg)^{1/2}\Bigg(\sum_{\substack{k_{p+1}\in\mathbb{Z}^d\\ |k_{p+1}|=\lambda_{n_{p+1}}}}\mathbf{1}_{\{|k_0-k_{p+1}|\leqslant\lambda_{n'}\}}\|\widetilde{\Pi}_{k_{p+1}}u_{p+1}\|_{L^2}^2\Bigg)^{1/2}.\end{aligned} \tag{5.17}$$

By Lemma 4.3, we have

$$\sum_{\substack{k_{p+1}\in\mathbb{Z}^d\\ |k_{p+1}|=\lambda_{n_{p+1}}}}\mathbf{1}_{\{|k_0-k_{p+1}|\leqslant\lambda_{n'}\}}\leqslant C\sum_{\substack{k_{p+1}\in\mathbb{Z}^d\\ |k_{p+1}|=\lambda_{n_{p+1}}}}\frac{(1+\lambda_{n'})^{d+1}}{(|k_0-k_{p+1}|+1+\lambda_{n'})^{d+1}}\leqslant C(1+\lambda_{n'})^{d}. \tag{5.18}$$

Thus, by the above analysis and (5.18), we finally obtain

$$\begin{aligned}&\big\|\Pi_{n_0}M(\Pi_{n_1}u_1,\ldots,\Pi_{n_{p+1}}u_{p+1})\big\|_{L^2}^2\\ &\quad\leqslant C(1+\lambda_{n_0}+\lambda_{n_{p+1}})^{2\tau}(1+\lambda_{n'})^{2\nu+d}\prod_{j=1}^{p}\|u_j\|_{L^2}^2\\ &\qquad\times\sum_{\substack{k_0,k_{p+1}\in\mathbb{Z}^d\\ |k_0|=\lambda_{n_0},\,|k_{p+1}|=\lambda_{n_{p+1}}}}\mathbf{1}_{\{|k_0-k_{p+1}|\leqslant\lambda_{n'}\}}\|\widetilde{\Pi}_{k_{p+1}}u_{p+1}\|_{L^2}^2\\ &\quad\leqslant C(1+\lambda_{n_0}+\lambda_{n_{p+1}})^{2\tau}(1+\lambda_{n'})^{2\nu+2d}\prod_{j=1}^{p}\|u_j\|_{L^2}^2\sum_{\substack{k_{p+1}\in\mathbb{Z}^d\\ |k_{p+1}|=\lambda_{n_{p+1}}}}\|\widetilde{\Pi}_{k_{p+1}}u_{p+1}\|_{L^2}^2\\ &\quad= C(1+\lambda_{n_0}+\lambda_{n_{p+1}})^{2\tau}(1+\lambda_{n'})^{2\nu+2d}\prod_{j=1}^{p}\|u_j\|_{L^2}^2\|\Pi_{n_{p+1}}u_{p+1}\|_{L^2}^2\\ &\quad\leqslant C(1+\lambda_{n_0}+\lambda_{n_{p+1}})^{2\tau}(1+\lambda_{n'})^{2\nu+2d}\prod_{j=1}^{p+1}\|u_j\|_{L^2}^2.\end{aligned} \tag{5.19}$$

Thus, (5.2) holds true with ν replaced with $\nu+d$. This concludes the proof. □

We have seen in Proposition 5.2 that any element in $\mathcal{M}^{\nu,\tau}_{p+1,\delta}$ maps $H^s(\mathbb{T}^d) \times \cdots \times H^s(\mathbb{T}^d)$ to $H^{s-\tau-1}(\mathbb{T}^d)$. But the elements in its subspace $\mathcal{G}^{\nu',\tau}_{p+1,\delta}$ have a better property. As a matter of fact, if we mimic the proof of Proposition 5.2 but using (5.12) and Lemma 4.3 instead of using (5.2) and Lemma 4.1, then we get

Proposition 5.6. *Let $\nu \in \mathbb{R}_+$, $\tau \in \mathbb{R}$, $\delta \in (0,1)$, $p \in \mathbb{N}^*$, $s > \nu + d + 1$. Then any element $M \in \mathcal{G}^{\nu,\tau}_{p+1,\delta}$ extends as a bounded operator from $H^s(\mathbb{T}^d) \times \cdots \times H^s(\mathbb{T}^d)$ to $H^{s-\tau}(\mathbb{T}^d)$. Moreover, for any $s_0 \in (\nu + d + 1, s]$, there is $C > 0$ such that for any $M \in \mathcal{G}^{\nu,\tau}_{p+1,\delta}$ and any $u_1, \ldots, u_{p+1} \in H^s(\mathbb{T}^d)$,*

$$\left\| M(u_1, \ldots, u_{p+1}) \right\|_{H^{s-\tau}} \leqslant C \|M\|_{\mathcal{G}^{\nu,\tau}_{p+1,\delta}} \prod_{j=1}^{p} \|u_j\|_{H^{s_0}} \|u_{p+1}\|_{H^s}. \tag{5.20}$$

We shall also need classes of remainder operators. If $n_1, \ldots, n_{p+1} \in \mathbb{N}$ and if $j_0 \in \{1, \ldots, p+1\}$ is an index such that $\lambda_{n_{j_0}} = \max\{\lambda_{n_1}, \ldots, \lambda_{n_{p+1}}\}$, where λ_{n_j}, $j = 1, \ldots, p+1$, are defined by (3.4), we denote

$$\max{}_2(\lambda_{n_1}, \ldots, \lambda_{n_{p+1}}) = 1 + \max\{\lambda_{n_j};\ 1 \leqslant j \leqslant p+1,\ j \neq j_0\}. \tag{5.21}$$

Definition 5.7. Let $\nu \in \mathbb{R}_+$, $\tau \in \mathbb{R}$, $p \in \mathbb{N}^*$. We denote by $\mathcal{R}^{\nu,\tau}_{p+1}$ the space of $(p+1)$-linear maps from $\mathcal{E} \times \cdots \times \mathcal{E}$ to $L^2(\mathbb{T}^d)$, $(u_1, \ldots, u_{p+1}) \to R(u_1, \ldots, u_{p+1})$ such that for any $N \in \mathbb{R}_+$, there is $C_N > 0$ such that for any $(n_0, \ldots, n_{p+1}) \in \mathbb{N}^{p+2}$, any $u_1, \ldots, u_{p+1} \in \mathcal{E}$,

$$\left\| \Pi_{n_0} R(\Pi_{n_1} u_1, \ldots, \Pi_{n_{p+1}} u_{p+1}) \right\|_{L^2} \leqslant C_N (1 + \lambda_{n_0})^{\tau} \frac{\max_2(\lambda_{n_1}, \ldots, \lambda_{n_{p+1}})^{\nu+N}}{(\lambda_{n_0} + \cdots + \lambda_{n_{p+1}} + 1)^N} \prod_{j=1}^{p+1} \|u_j\|_{L^2}. \tag{5.22}$$

The elements in $\mathcal{R}^{\nu,\tau}_{p+1}$ also extend as bounded operators on Sobolev spaces.

Proposition 5.8. *Let $\nu \in \mathbb{R}_+$, $\tau \in \mathbb{R}$, $p \in \mathbb{N}^*$ be given. Then for any $s, s_0 \in \mathbb{R}$ with $s \geqslant s_0 > 2$, any $R \in \mathcal{R}^{\nu,\tau}_{p+1}$, $(u_1, \ldots, u_{p+1}) \to R(u_1, \ldots, u_{p+1})$ extends as a bounded map from $H^s(\mathbb{T}^d) \times \cdots \times H^s(\mathbb{T}^d) \to H^{2s-\nu-\tau-7}(\mathbb{T}^d)$. Moreover we have*

$$\left\| R(u_1, \ldots, u_{p+1}) \right\|_{H^{2s-\nu-\tau-7}} \leqslant C \sum_{1 \leqslant j_1 < j_2 \leqslant p+1} \left[\|u_{j_1}\|_{H^s} \|u_{j_2}\|_{H^s} \prod_{k \neq j_1, k \neq j_2} \|u_k\|_{H^{s_0}} \right]. \tag{5.23}$$

Remark 5.9. In fact, any $R \in \mathcal{R}^{\nu,\tau}_{p+1}$, $(u_1, \ldots, u_{p+1}) \to R(u_1, \ldots, u_{p+1})$ extends as a bounded map from $H^s(\mathbb{T}^d) \times \cdots \times H^s(\mathbb{T}^d) \to H^{2s-\nu-\tau-a}(\mathbb{T}^d)$ for any $a > 6$ and we also have a counterpart of (5.23). We take $a = 7$ for the convenience of expression and this will be enough for the use.

Proof. We may assume $\tau = 0$. We need to bound $\|\Pi_{n_0} R(u_1, \ldots, u_{p+1})\|_{L^2}$ from above by $(1 + \lambda_{n_0})^{-2s+\nu+7} c_{n_0}$ for an ℓ^2-sequence c_{n_0}. To do that we decompose u_j as $\sum_{n_j} \Pi_{n_j} u_j$ and use (5.22). By the symmetries we limit ourselves to summation over

$$n_1 \leqslant \cdots \leqslant n_{p+1}, \tag{5.24}$$

which according to (i) of Lemma 4.1 is equivalent to

$$\lambda_{n_1} \leqslant \cdots \leqslant \lambda_{n_{p+1}}. \tag{5.25}$$

From this we get

$$\max_2(\lambda_{n_1},\dots,\lambda_{n_{p+1}}) = 1+\lambda_{n_p}. \tag{5.26}$$

Therefore we are done if we can bound from above

$$C\sum_{n_1\leqslant\cdots\leqslant n_{p+1}}\frac{(1+\lambda_{n_p})^{\nu+N}}{(1+\lambda_{n_0}+\cdots+\lambda_{n_{p+1}})^N}\prod_{j=1}^{p-1}(1+\lambda_{n_j})^{-s_0}(1+\lambda_{n_p})^{-s}(1+\lambda_{n_{p+1}})^{-s} \tag{5.27}$$

by $(1+\lambda_{n_0})^{-2s+\nu+7}c_{n_0}$ for $s\geqslant 0$, since $\|\Pi_{n_j}u_j\|_{L^2}\leqslant(1+\lambda_{n_j})^{-\rho}\|u_j\|_{H^\rho}$ holds true with any $\rho>0$. Using (5.25) we bound (5.27) from above by

$$C\sum_{n_1\leqslant\cdots\leqslant n_{p+1}}\frac{(1+\lambda_{n_p})^{\nu+N-2s}}{(1+\lambda_{n_0}+\lambda_{n_{p+1}})^N}\prod_{j=1}^{p-1}(1+\lambda_{n_j})^{-s_0}. \tag{5.28}$$

Applying (iv) of Lemma 4.1 to n_{p+1} sum, we see that (5.28) is not larger than

$$C\sum_{n_1\leqslant\cdots\leqslant n_p}(1+\lambda_{n_p})^{\nu+N-2s}(1+\lambda_{n_0})^{-(N-2)}\prod_{j=1}^{p-1}(1+\lambda_{n_j})^{-s_0}.$$

Also because of (iv) of Lemma 4.1, this can be bounded from above by $(1+\lambda_{n_0})^{-2s+\nu+7}c_{n_0}$ with $(c_{n_0})_{n\in\mathbb{N}}$ an ℓ^2-sequence if we take $N=2s-\nu-5/2$, $s_0>2$ and thus concludes the proof. $\square$

Definition 5.10. Let $\nu\in\mathbb{R}_+$, $\tau\in\mathbb{R}$, $p\in\mathbb{N}^*$, $\omega:\{0,\dots,p+1\}\to\{-1,1\}$ be given.

- If $\sum_{j=0}^{p+1}\omega(j)\neq 0$, we set $\widetilde{\mathcal{R}}^{\nu,\tau}_{p+1}(\omega)=\mathcal{R}^{\nu,\tau}_{p+1}$;
- If $\sum_{j=0}^{p+1}\omega(j)=0$, we denote by $\widetilde{\mathcal{R}}^{\nu,\tau}_{p+1}(\omega)$ the closed subspace of $\mathcal{R}^{\nu,\tau}_{p+1}$ given by those $R\in\mathcal{R}^{\nu,\tau}_{p+1}$ such that

$$\Pi_{n_0}R(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})\equiv 0 \tag{5.29}$$

for any $(n_0,\dots,n_{p+1})\in S^\omega_p$, where S^ω_p is defined by (5.10).

6. Rewriting of the energy

We shall finally control the energy. But in this section let us compute its time derivative. We shall write it in terms of several types of multilinear operators introduced in the previous section according to the estimate of the function of type (1.4). One difficulty is to make appear a commutator so that we can gain one derivative to obtain longer lifespan than the one given by local existence theory. There is another thing we should take care of. In order to recover the loss of derivatives coming from dividing small divisors, we shall use normal forms only to eliminate the low frequency part of one type of multilinear operators (in fact these are M^p_ℓ) and we have to properly estimate the high frequency part. Since by Proposition 5.2 there is one derivative loss when we extend the operators in $\mathcal{M}^{\nu,\tau}_{p+1,\delta}$ to Sobolev spaces, we have to exploit a better property of the high frequency part in order to not lose derivatives. This is realized by showing $M^p_\ell\in\mathcal{G}^{\nu,2s-2}_{p+1,\delta}$.

We begin by analyzing the nonlinearity. Using Taylor's formula we have

$$-F(v) = -\sum_{p=\kappa}^{2\kappa-1} \frac{\partial_v^{p+1} F(0)}{(p+1)!} v^{p+1} + G(v) \tag{6.1}$$

where $G(v)$ vanishes at order $2\kappa + 1$ at $v = 0$. (Here we only decompose the nonlinearity up to order 2κ because it is enough for us to get a lifespan of length $c\epsilon^{-3/2}$.) By making a change of unknown

$$u = (D_t + \Lambda_m)v, \qquad v = \frac{1}{2}\Lambda_m^{-1}(u + \bar{u}) \tag{6.2}$$

with

$$D_t = -i\partial_t, \qquad \Lambda_m = \sqrt{-\Delta + m^2}, \tag{6.3}$$

we write Cauchy problem (2.2) as

$$\begin{aligned} (D_t - \Lambda_m)u &= -F\left(\Lambda_m^{-1}\left(\frac{u + \bar{u}}{2}\right)\right), \\ u|_{t=0} &= \epsilon u_0 \end{aligned} \tag{6.4}$$

with $u_0 = -iv_1 + \Lambda_m v_0 \in H^s(\mathbb{T}^d, \mathbb{C})$. The energy to be estimated is

$$\Theta_s\big(u(t,\cdot)\big) = \frac{1}{2}\big\langle \Lambda_m^s u(t,\cdot), \Lambda_m^s u(t,\cdot)\big\rangle. \tag{6.5}$$

Proposition 6.1. *There are $\nu \in \mathbb{R}_+$, $\delta \in (0,1)$ and large enough s_0 such that for any $s \geqslant s_0$, there are:*

- *Multilinear operators $M_\ell^p \in \mathcal{G}_{p+1,\delta}^{\nu,2s-2} \cap \widetilde{\mathcal{M}}_{p+1,\delta}^{\nu+d,2s-2}(\omega_\ell)$, $\kappa \leqslant p \leqslant 2\kappa - 1$, $0 \leqslant \ell \leqslant p$ with ω_ℓ defined by $\omega_\ell(j) = -1$ for $j = 0, \ldots, \ell$, and $\omega_\ell(j) = 1$ for $j = \ell + 1, \ldots, p + 1$;*
- *Multilinear operators $\widetilde{M}_\ell^p \in \widetilde{\mathcal{M}}_{p+1,\delta}^{\nu,2s-1}(\widetilde{\omega}_\ell)$, $\kappa \leqslant p \leqslant 2\kappa - 1$, $0 \leqslant \ell \leqslant p$ with $\widetilde{\omega}_\ell$ defined by $\widetilde{\omega}_\ell(j) = -1$ for $j = 0, \ldots, \ell, p + 1$, and $\widetilde{\omega}_\ell(j) = 1$ for $j = \ell + 1, \ldots, p$;*
- *Multilinear operators $R_\ell^p \in \widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\omega_\ell)$, $\widetilde{R}_\ell^p \in \widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\widetilde{\omega}_\ell)$, $\kappa \leqslant p \leqslant 2\kappa - 1$, $0 \leqslant \ell \leqslant p$;*
- *A map $u \to T(u)$ defined on $H^s(\mathbb{T}^d)$ with values in $\mathbb{R}$, satisfying when $\|u\|_{H^s} \leqslant 1$, $|T(u)| \leqslant C\|u\|_{H^s}^{2\kappa+2}$*

such that

$$\begin{aligned} \frac{d}{dt}\Theta_s\big(u(t,\cdot)\big) &= \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} \operatorname{Re} i\big\langle M_\ell^p(\underbrace{\bar{u},\ldots,\bar{u}}_{\ell}, \underbrace{u,\ldots,u}_{p+1-\ell}), u\big\rangle \\ &\quad + \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} \operatorname{Re} i\big\langle \widetilde{M}_\ell^p(\underbrace{\bar{u},\ldots,\bar{u}}_{\ell}, \underbrace{u,\ldots,u}_{p-\ell}, \bar{u}), u\big\rangle \\ &\quad + \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} \operatorname{Re} i\big\langle R_\ell^p(\underbrace{\bar{u},\ldots,\bar{u}}_{\ell}, \underbrace{u,\ldots,u}_{p+1-\ell}), u\big\rangle \end{aligned}$$

$$+\sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\, i\big\langle \widetilde{R}_\ell^p(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p-\ell},\bar{u}),u\big\rangle$$
$$+\,T(u). \tag{6.6}$$

Remark 6.2. Let us explain the meaning of the proposition. In the right-hand side of (6.6), we have two main contributions: the M_ℓ^p terms will be expressed in the proof below from commutators. This will make gain one derivative, and explains why these terms are of order $2s-2$, and not just $2s-1$. On the other hand, the $\widetilde{M}_\ell^p$ terms are of order $2s-1$ because their expression will not involve any commutator.

In the rest of the paper, we shall modify Θ_s in the left-hand side of (6.6) in order to cancel out the M_ℓ^p, $\widetilde{M}_\ell^p$ terms in the right-hand side. The fact that we gained two derivatives on M_ℓ^p will allow us to loose one derivative when constructing the corrector used to eliminate this term. On the other hand, we shall see that $\widetilde{M}_\ell^p$ is a "noncharacteristic" term, which can be eliminated without loosing derivatives (actually we shall even gain one derivative): see Lemma 8.1. This explains why the fact that $\widetilde{M}_\ell^p$ is only of order $2s-1$ is unimportant.

Proof. We compute according to (6.1) and (6.4)

$$\begin{aligned}
\frac{d}{dt}\Theta_s\big(u(t,\cdot)\big) &= Re\, i\big\langle \Lambda_m^s D_t u,\, \Lambda_m^s u\big\rangle = Re\, i\left\langle -\Lambda_m^s F\left(\Lambda_m^{-1}\left(\frac{u+\bar{u}}{2}\right)\right),\, \Lambda_m^s u\right\rangle\\
&= \sum_{p=\kappa}^{2\kappa-1} c(p)\, Re\, i\left\langle \Lambda_m^s\left(\Lambda_m^{-1}\left(\frac{u+\bar{u}}{2}\right)\right)^p \Lambda_m^{-1}\bar{u},\, \Lambda_m^s u\right\rangle\\
&\quad+ \sum_{p=\kappa}^{2\kappa-1} c(p)\, Re\, i\left\langle \Lambda_m^s\left(\Lambda_m^{-1}\left(\frac{u+\bar{u}}{2}\right)\right)^p \Lambda_m^{-1}u,\, \Lambda_m^s u\right\rangle\\
&\quad+ Re\, i\left\langle \Lambda_m^s G\left(\Lambda_m^{-1}\left(\frac{u+\bar{u}}{2}\right)\right),\, \Lambda_m^s u\right\rangle,
\end{aligned} \tag{6.7}$$

where $c(p)$ is a real constant. The last term in the right-hand side of (6.7) contributes to the last term $T(u)$ in (6.6) by Lemma 4.2 if s is large. We then have to compute I and II with

$$\begin{aligned}
\mathrm{I} &= c(p)\, Re\, i\left\langle \Lambda_m^s\left(\Lambda_m^{-1}\left(\frac{u+\bar{u}}{2}\right)\right)^p \Lambda_m^{-1}\bar{u},\, \Lambda_m^s u\right\rangle,\\
\mathrm{II} &= c(p)\, Re\, i\left\langle \Lambda_m^s\left(\Lambda_m^{-1}\left(\frac{u+\bar{u}}{2}\right)\right)^p \Lambda_m^{-1}u,\, \Lambda_m^s u\right\rangle.
\end{aligned} \tag{6.8}$$

This is the content of the next several lemmas.

Lemma 6.3. *There are $\nu \in \mathbb{R}_+$ and small enough δ such that there are multilinear operators $\widetilde{M}_\ell^p \in \widetilde{\mathcal{M}}_{p+1,\delta}^{\nu,2s-1}(\widetilde{\omega}_\ell)$, $\widetilde{R}_\ell^p \in \widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\widetilde{\omega}_\ell)$ with $\kappa \leqslant p \leqslant 2\kappa-1$, $0 \leqslant \ell \leqslant p$ and $\widetilde{\omega}_\ell$ defined by $\widetilde{\omega}_\ell(j)=-1$ for $j=0,\dots,\ell$, $\widetilde{\omega}_\ell(j)=1$ for $j=\ell+1,\dots,p$ and $\widetilde{\omega}_\ell(p+1)=-1$ such that*

$$I=\sum_{\ell=0}^{p} Re\, i\big\langle \widetilde{M}_\ell^p(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p-\ell},\bar{u}),u\big\rangle+\sum_{\ell=0}^{p} Re\, i\big\langle \widetilde{R}_\ell^p(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p-\ell},\bar{u}),u\big\rangle. \tag{6.9}$$

Proof. Let $\widetilde{\omega}_\ell$ be defined as in the statement of the lemma and $S_p^{\widetilde{\omega}_\ell}$ defined by (5.10) with ω replaced by $\widetilde{\omega}_\ell$. Let $0 < \delta \ll 1$. We shall use notation (3.6) in the following. We decompose

$$\mathrm{I} = c(p)\, Re\, i\big\langle A_p^1(\bar{u}, u)\bar{u}, u\big\rangle + c(p)\, Re\, i\big\langle A_p^2(\bar{u}, u)\bar{u}, u\big\rangle, \tag{6.10}$$

where A_p^i, $i = 1, 2$, are operators defined as follows:

$$A_p^1(\bar{u}, u) = \sum_{n_1,\dots,n_{p+1}\in\mathbb{N}} \mathbf{1}_{\{\lambda_{n'} < \delta\lambda_{n_{p+1}}\}} \Lambda_m^{2s} \prod_{j=1}^{p} \left(\Lambda_m^{-1} \Pi_{n_j} \left(\frac{\bar{u} + u}{2} \right) \right) \Lambda_m^{-1} \Pi_{n_{p+1}}, \tag{6.11}$$

$$A_p^2(\bar{u}, u) = \sum_{n_1,\dots,n_{p+1}\in\mathbb{N}} \mathbf{1}_{\{\lambda_{n'} \geqslant \delta\lambda_{n_{p+1}}\}} \Lambda_m^{2s} \prod_{j=1}^{p} \left(\Lambda_m^{-1} \Pi_{n_j} \left(\frac{\bar{u} + u}{2} \right) \right) \Lambda_m^{-1} \Pi_{n_{p+1}}. \tag{6.12}$$

We denote

$$\mathbf{a}_\ell(\tilde{n}; u_1, \dots, u_p) = \frac{c(p)}{2^p} \binom{p}{\ell} \big(\Lambda_m^{-1} \Pi_{n_1} u_1 \big) \cdots \big(\Lambda_m^{-1} \Pi_{n_p} u_p \big) \tag{6.13}$$

and define

$$\widetilde{M}_\ell^p(u_1, \dots, u_{p+1}) = \sum_{\vec{n}\in(S_p^{\widetilde{\omega}})^c} \mathbf{1}_{\{\lambda_{n'} < \delta\lambda_{n_{p+1}}\}} \Lambda_m^{2s} \Pi_{n_0} \mathbf{a}_\ell(\tilde{n}; u_1, \dots, u_p) \Lambda_m^{-1} \Pi_{n_{p+1}} u_{p+1},$$

$$\widetilde{R}_\ell^p(u_1, \dots, u_{p+1}) = \sum_{\vec{n}\in(S_p^{\widetilde{\omega}})^c} \mathbf{1}_{\{\lambda_{n'} \geqslant \delta\lambda_{n_{p+1}}\}} \Lambda_m^{2s} \Pi_{n_0}\, \mathbf{a}_\ell(\tilde{n}; u_1, \dots, u_p) \Lambda_m^{-1} \Pi_{n_{p+1}} u_{p+1}. \tag{6.14}$$

Then we claim that

$$Re\, i\big\langle \widetilde{M}_\ell^p (\underbrace{\bar{u}, \dots, \bar{u}}_{\ell}, \underbrace{u, \dots, u}_{p-\ell}, \bar{u}), u\big\rangle$$

equals the quantity obtained replacing $\widetilde{M}_\ell^p$ by the expression given by the first equality in (6.14) but with a sum taken for all $n_0, \dots, n_{p+1}$. In fact, if $(n_0, \dots, n_{p+1}) \in S_p^{\widetilde{\omega}_\ell}$ with $S_p^{\widetilde{\omega}_\ell} \neq \emptyset$, then by definition there exists a bijection σ mapping $\{0, \dots, \ell, p+1\}$ to $\{\ell+1, \dots, p\}$ with $n_j = n_{\sigma(j)}$ for j in the first set. Therefore if we denote by $\widetilde{M}_\ell^{p,c}$ the multilinear operators obtained by the expression given by the first equality in (6.14) but with a sum for $\vec{n} \in S_p^{\widetilde{\omega}}$, by coupling $\Pi_{n_j}\bar{u}$, $j = 0, \dots, \ell, p+1$, respectively with $\Pi_{n_{\sigma(j)}} u$, $j = 0, \dots, \ell, p+1$, we deduce that

$$\big\langle \widetilde{M}_\ell^{p,c} (\underbrace{\bar{u}, \dots, \bar{u}}_{\ell}, \underbrace{u, \dots, u}_{p-\ell}, \bar{u}), u\big\rangle,$$

is real and thus contributes for nothing to the computation of I. With the same reasoning, if $\widetilde{R}_\ell^{p,c}$ denote the multilinear operators obtained by the expression given by the second equality in (6.14) but with a sum for $\vec{n} \in S_p^{\widetilde{\omega}}$, we see that

$$\big\langle \widetilde{R}_\ell^{p,c} (\underbrace{\bar{u}, \dots, \bar{u}}_{\ell}, \underbrace{u, \dots, u}_{p-\ell}, \bar{u}), u\big\rangle,$$

is real and thus also contributes for nothing to the computation of I. Therefore, (6.9) follows from the binomial expansion and the symmetries on $n_1, \dots, n_p$.

We are left to show that $\widetilde{M}_\ell^p \in \widetilde{\mathcal{M}}_{p+1,\delta}^{\nu,2s-1}(\widetilde{\omega}_\ell)$ and $\widetilde{R}_\ell^p \in \widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\widetilde{\omega}_\ell)$.

Assume that $\Pi_{n_0}\widetilde{M}_\ell^p(\Pi_{n_1}u_1, \dots, \Pi_{n_{p+1}}u_{p+1})$ is nonzero. Then by (ii) of Lemma 4.5 and the definition we have

$$|\lambda_{n_0} - \lambda_{n_{p+1}}| \leqslant \lambda_{n'}, \tag{6.15}$$

which, together with the cut-off function in the definition, implies

$$\lambda_{n'} < \delta\lambda_{n_{p+1}}, \qquad |\lambda_{n_0} - \lambda_{n_{p+1}}| < \frac{1}{2}(\lambda_{n_0} + \lambda_{n_{p+1}}) \tag{6.16}$$

if δ is small enough. From (6.15) and (6.16) we see that $\widetilde{M}_\ell^p$ satisfies item (i) of Definition 5.1. When we establish (5.2) for $\widetilde{M}_\ell^p$, we assume that the left-hand side of (5.2) with M replaced by $\widetilde{M}_\ell^p$ does not vanish. Then since we have (6.15) and (6.16), we deduce that λ_{n_0} and $\lambda_{n_{p+1}}$ are comparable to each other. Therefore, we obtain by (iii) of Lemma 4.5 that (5.2) holds true for any large enough $\nu \in \mathbb{R}_+$ and for $\tau = 2s - 1$, since there is Λ_m^{2s} before Π_{n_0} and Λ_m^{-1} before $\Pi_{n_{p+1}}$ in the definition of $\widetilde{M}_\ell^p$. Thus $\widetilde{M}_\ell^p \in \mathcal{M}_{p+1,\delta}^{\nu,2s-1}$. On the other hand, (5.9) holds true for $(n_0, \dots, n_{p+1}) \in S_p^{\widetilde{\omega}_\ell}$ since in the definition we have ruled out the sum over the indices in that set. So we have $\widetilde{M}_\ell^p \in \widetilde{\mathcal{M}}_{p+1,\delta}^{\nu,2s-1}(\widetilde{\omega}_\ell)$.

To prove $\widetilde{R}_\ell^p \in \widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\widetilde{\omega_\ell})$, we assume that the left-hand side of (5.22) with R replaced by $\widetilde{R}_\ell^p$ is nonzero. Then we have by Lemma 4.5 and the definition of $\widetilde{R}_\ell^p$

$$|\lambda_{n_0} - \lambda_{n_{p+1}}| \leqslant \lambda_{n'} \tag{6.17}$$

and also obtain

$$\lambda_{n'} \geqslant \delta\lambda_{n_{p+1}}, \tag{6.18}$$

which is due to the cut-off function in the definition. From (6.17) and (6.18) we deduce

$$\lambda_{n'} \geqslant C\lambda_{n_0}. \tag{6.19}$$

Therefore we get

$$(1 + \lambda_{n'}) \sim (\lambda_{n_0} + \dots + \lambda_{n_{p+1}} + 1). \tag{6.20}$$

Thus (5.22) with R replaced by $\widetilde{R}_\ell^p$ and $\tau = 2s$ follows from (iii) of Lemma 4.5, (6.20) and the fact that

$$(1 + \lambda_{n'}) \leqslant C \max{}_2(\lambda_{n_1}, \dots, \lambda_{n_{p+1}}).$$

On the other hand, (5.29) holds trivially for $(n_0, \dots, n_{p+1}) \in S_p^{\widetilde{\omega}_\ell}$ by the definition. Thus we have $\widetilde{R}_\ell^p \in \widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\widetilde{\omega}_\ell)$. $\square$

Lemma 6.4. *There are $\nu \in \mathbb{R}_+$ and small enough δ such that there are multilinear operators $M_\ell^p \in \mathcal{G}_{p+1,\delta}^{\nu,2s-2} \cap \widetilde{\mathcal{M}}_{p+1,\delta}^{\nu+d,2s-2}(\omega_\ell)$, $R_\ell^p \in \widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\omega_\ell)$ with $\kappa \leqslant p \leqslant 2\kappa-1$, $0 \leqslant \ell \leqslant p$, and ω_ℓ defined by $\omega_\ell(j) = -1$ for $j = 0,\dots,\ell$ and $\omega_\ell(j) = 1$ for $j = \ell+1,\dots,p+1$, such that*

$$II = \sum_{\ell=0}^{p} Re\, i\big\langle M_\ell^p(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p+1-\ell}),u\big\rangle + \sum_{\ell=0}^{p} Re\, i\big\langle R_\ell^p(\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p+1-\ell}),u\big\rangle. \tag{6.21}$$

Proof. By the adjointness of the operator Λ_m and (6.8)

$$\mathrm{II} = \frac{c(p)}{2} Re\, i\left\langle\left(\Lambda_m^{2s}\left(\Lambda_m^{-1}\left(\frac{\bar{u}+u}{2}\right)\right)^p\Lambda_m^{-1} - \Lambda_m^{-1}\left(\Lambda_m^{-1}\left(\frac{\bar{u}+u}{2}\right)\right)^p\Lambda_m^{2s}\right)u,u\right\rangle. \tag{6.22}$$

Let $\delta > 0$ be small. We then define B_p^1, B_p^2 to be the operators

$$B_p^1(\bar{u},u) = \frac{c(p)}{2}\sum_{\vec{n}\in\mathbb{N}^{p+2}} \mathbf{1}_{\{\lambda_{n'}\geqslant\delta\lambda_{n_{p+1}}\}}\Pi_{n_0}\left(\prod_{j=1}^{p}\left(\Lambda_m^{-1}\Pi_{n_j}\left(\frac{u+\bar{u}}{2}\right)\right)\right)\Pi_{n_{p+1}}, \tag{6.23}$$

$$B_p^2(\bar{u},u) = \frac{c(p)}{2}\sum_{\vec{n}\in\mathbb{N}^{p+2}} \mathbf{1}_{\{\lambda_{n'}<\delta\lambda_{n_{p+1}}\}}\Pi_{n_0}\left(\prod_{j=1}^{p}\left(\Lambda_m^{-1}\Pi_{n_j}\left(\frac{u+\bar{u}}{2}\right)\right)\right)\Pi_{n_{p+1}}. \tag{6.24}$$

Remark that by the symmetries on $n_1,\dots,n_p$ and with notation (6.13), they may also be written

$$B_p^1(\bar{u},u) = \frac{1}{2}\sum_{\vec{n}\in\mathbb{N}^{p+2}}\sum_{\ell=0}^{p} \mathbf{1}_{\{\lambda_{n'}\geqslant\delta\lambda_{n_{p+1}}\}}\Pi_{n_0}\mathbf{a}_\ell(\tilde{n};\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p-\ell})\Pi_{n_{p+1}}, \tag{6.25}$$

$$B_p^2(\bar{u},u) = \frac{1}{2}\sum_{\vec{n}\in\mathbb{N}^{p+2}}\sum_{\ell=0}^{p} \mathbf{1}_{\{\lambda_{n'}<\delta\lambda_{n_{p+1}}\}}\Pi_{n_0}\mathbf{a}_\ell(\tilde{n};\underbrace{\bar{u},\dots,\bar{u}}_{\ell},\underbrace{u,\dots,u}_{p-\ell})\Pi_{n_{p+1}}. \tag{6.26}$$

Then we have by (6.22)

$$\begin{aligned}
\mathrm{II} &= Re\, i\big\langle\big(\Lambda_m^{2s}\big(B_p^1(\bar{u},u)+B_p^2(\bar{u},u)\big)\Lambda_m^{-1} - \Lambda_m^{-1}\big(B_p^1(\bar{u},u)+B_p^2(\bar{u},u)\big)\Lambda_m^{2s}\big)u,u\big\rangle \\
&= Re\, i\big\langle\Lambda_m^{2s}B_p^1(\bar{u},u)\Lambda_m^{-1}u,u\big\rangle - Re\, i\big\langle\Lambda_m^{-1}B_p^1(\bar{u},u)\Lambda_m^{2s}u,u\big\rangle \\
&\quad + \big\langle\big[\Lambda_m^{2s},B_p^2(\bar{u},u)\big]\Lambda_m^{-1}u,u\big\rangle - Re\, i\big\langle\big[\Lambda_m^{-1},B_p^2(\bar{u},u)\big]\Lambda_m^{2s}u,u\big\rangle \\
&:= \mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI}.
\end{aligned} \tag{6.27}$$

Let ω_ℓ be defined by $\omega_\ell(j) = -1$ for $j = 0,\dots,\ell$, and $\omega_\ell(j) = 1$ for $j = \ell+1,\dots,p+1$. Let $S_p^{\omega_\ell}$ be defined by (5.10) with ω replaced by ω_ℓ.

(i) The term III.

With notations (3.6) and (6.13) we set

$$R_\ell^{p,1}(u_1,\dots,u_{p+1}) = \frac{1}{2}\sum_{\vec{n}\in(S_p^{\omega_\ell})^c} \mathbf{1}_{\{\lambda_{n'}\geqslant\delta\lambda_{n_{p+1}}\}}\Lambda_m^{2s}\Pi_{n_0}\mathbf{a}_\ell(\tilde{n};u_1,\dots,u_p)\Lambda_m^{-1}\Pi_{n_{p+1}}u_{p+1} \tag{6.28}$$

so that

$$\mathrm{III} = \sum_{\ell=0}^{p} Re\, i\big\langle R_\ell^{p,1}(\underbrace{\bar u, \ldots, \bar u}_{\ell}, \underbrace{u, \ldots, u}_{p+1-\ell}), u\big\rangle. \tag{6.29}$$

Note that we have ruled out the indices $(n_0, \ldots, n_{p+1}) \in S_p^{\omega_\ell}$ in the definition with the same reasoning as in the proof of Lemma 6.3. Therefore, (5.29) with R replaced with $R_\ell^{p,1}$ holds trivially for $(n_0, \ldots, n_{p+1}) \in S_p^{\omega_\ell}$. We now assume that the left-hand side of (5.22) with R replaced by $R_\ell^{p,1}$ is nonzero. Then by Lemma 4.5 and the definition we must have

$$\lambda_{n'} \geqslant \delta\lambda_{n_{p+1}}, \qquad |\lambda_{n_0} - \lambda_{n_{p+1}}| \leqslant \lambda_{n'}, \tag{6.30}$$

which implies

$$\lambda_{n'} \geqslant C\lambda_{n_0}, \qquad (1+\lambda_{n'}) \sim (\lambda_{n_0} + \cdots + \lambda_{n_{p+1}} + 1). \tag{6.31}$$

On the other hand, we always have

$$1 + \lambda_{n'} \leqslant C \max{}_2(\lambda_{n_1}, \ldots, \lambda_{n_{p+1}}). \tag{6.32}$$

Thus, by Lemma 4.5 and the above inequalities we see that (5.22) holds with $\tau = 2s$ and R replaced by $R_\ell^{p,1}$. So we get $R_\ell^{p,1} \in \widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\omega_\ell)$.

(ii) The term IV.

By (6.23) and using the fact $\overline{\Pi_{n_j} w} = \Pi_{n_j}\bar w$ we easily deduce

$$B_p^1(\bar u, u)^* = \frac{c(p)}{2} \sum_{\vec n \in \mathbb{N}^{p+2}} \mathbf{1}_{\{\lambda_{n'} \geqslant \delta\lambda_{n_0}\}} \Pi_{n_0} \left(\prod_{j=1}^{p} \left(\Lambda_m^{-1} \Pi_{n_j} \left(\frac{\bar u + u}{2} \right) \right) \right) \Pi_{n_{p+1}}, \tag{6.33}$$

which, using the symmetries on $(n_1, \ldots, n_p)$, may also be written

$$B_p^1(\bar u, u)^* = \frac{1}{2} \sum_{\ell=0}^{p} \sum_{\vec n \in \mathbb{N}^{p+2}} \mathbf{1}_{\{\lambda_{n'} \geqslant \delta\lambda_{n_0}\}} \Pi_{n_0} \mathbf{a}_\ell(\tilde n; \underbrace{\bar u, \ldots, \bar u}_{\ell}, \underbrace{u, \ldots, u}_{p-\ell}) \Pi_{n_{p+1}}. \tag{6.34}$$

Set

$$R_\ell^{p,2}(u_1, \ldots, u_{p+1}) = \frac{1}{2} \sum_{\vec n \in (S_p^{\omega_\ell})^c} \mathbf{1}_{\{\lambda_{n'} \geqslant \delta\lambda_{n_0}\}} \Lambda_m^{2s} \Pi_{n_0} \mathbf{a}_\ell(\tilde n; u_1, \ldots, u_p) \Lambda_m^{-1} \Pi_{n_{p+1}} u_{p+1} \tag{6.35}$$

so that

$$\mathrm{IV} = Re\, i\big\langle \Lambda_m^{2s} B_p^1(\bar u, u)^* \Lambda_m^{-1} u, u\big\rangle = \sum_{\ell=0}^{p} Re\, i\big\langle R_\ell^{p,2}(\underbrace{\bar u, \ldots, \bar u}_{\ell}, \underbrace{u, \ldots, u}_{p+1-\ell}), u\big\rangle, \tag{6.36}$$

provided we rule out the sum over the indices $(n_0, \ldots, n_{p+1}) \in S_p^{\omega_\ell}$, which may be shown with the same reasoning as in the proof of Lemma 6.3. Now if we assume that the left-hand side of (5.22) with R replaced by $R_\ell^{p,2}$ is nonzero, we also have (6.30)–(6.32). Thus by Lemma 4.5 we see that (5.22) with R replaced by $R_\ell^{p,2}$ holds. Therefore $R_\ell^{p,2} \in \mathcal{R}_{p+1}^{\nu,2s}$. Since (5.29) with R replaced by $R_\ell^{p,2}$ is trivial for $(n_0,\dots,n_{p+1}) \in S_p^{\omega_\ell}$, $R_\ell^{p,2} \in \widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\omega_\ell)$.

(iii) The term V.

First we claim that if δ is small, we have

$$B_p^2(\bar u,u) = \sum_{k_0,k_{p+1}\in\mathbb{Z}^d} \mathbf{1}_{\{|k_0-k_{p+1}|<\frac12(|k_0|+|k_{p+1}|)\}} \widetilde{\Pi}_{k_0} B_p^2(\bar u,u)\widetilde{\Pi}_{k_{p+1}}. \tag{6.37}$$

Indeed, we only need to show that $\widetilde{B}_p^2(\bar u,u)$ defined by

$$\widetilde{B}_p^2(\bar u,u) = \sum_{k_0,k_{p+1}\in\mathbb{Z}^d} \mathbf{1}_{\{|k_0-k_{p+1}|\geqslant\frac12(|k_0|+|k_{p+1}|)\}} \widetilde{\Pi}_{k_0} B_p^2(\bar u,u)\widetilde{\Pi}_{k_{p+1}}, \tag{6.38}$$

is a zero operator. By (6.26) we decompose

$$\begin{aligned}\widetilde{B}_p^2(\bar u,u) = \frac12 &\sum_{k_0,k_{p+1}\in\mathbb{Z}^d} \sum_{\tilde n\in\mathbb{N}^{p+2}} \sum_{\ell=0}^{p} \mathbf{1}_{\{\lambda_{n'}<\delta\lambda_{n_{p+1}},\ |k_0-k_{p+1}|\geqslant\frac12(|k_0|+|k_{p+1}|)\}} \\ &\times \widetilde{\Pi}_{k_0}\Pi_{n_0}\mathfrak{a}_\ell(\tilde n;\underbrace{\bar u,\dots,\bar u}_{\ell},\underbrace{u,\dots,u}_{p-\ell})\Pi_{n_{p+1}}\widetilde{\Pi}_{k_{p+1}}.\end{aligned} \tag{6.39}$$

From the facts that $\widetilde{\Pi}_{k_j}\Pi_{n_j} = \Pi_{n_j}\widetilde{\Pi}_{k_j}$, $j=0,p+1$ and that $\widetilde{\Pi}_{k_j}\Pi_{n_j}\neq 0$ if and only if $|k_j|=\lambda_{n_j}$ for $j=0,p+1$ and Lemma 4.5, we see that $\widetilde{\Pi}_{k_0}\Pi_{n_0}\mathfrak{a}_\ell(\tilde n;\underbrace{\bar u,\dots,\bar u}_{\ell},\underbrace{u,\dots,u}_{p-\ell})\Pi_{n_{p+1}}\widetilde{\Pi}_{k_{p+1}}$ is supported on

$$|k_0|=\lambda_{n_0},\qquad |k_{p+1}|=\lambda_{n_{p+1}},\qquad |k_0-k_{p+1}|\leqslant\lambda_{n'},\qquad |\lambda_{n_0}-\lambda_{n_{p+1}}|\leqslant\lambda_{n'}, \tag{6.40}$$

which is not compatible with

$$\lambda_{n'}<\delta\lambda_{n_{p+1}},\qquad |k_0-k_{p+1}|\geqslant\frac12\big(|k_0|+|k_{p+1}|\big), \tag{6.41}$$

if $\delta<1/2$. Thus $\widetilde{B}_p^2(\bar u,u)=0$.

By (6.26) and (6.39) we set

$$\begin{aligned}M_\ell^{p,1}(u_1,\dots,u_{p+1}) = \frac12 &\sum_{k_0,k_{p+1}\in\mathbb{Z}^d} \sum_{\tilde n\in(S_p^{\omega_\ell})^c} \mathbf{1}_{\{\lambda_{n'}<\delta\lambda_{n_{p+1}},\ |k_0-k_{p+1}|<\frac12(|k_0|+|k_{p+1}|)\}} \\ &\times\big(\big(m^2+|k_0|^2\big)^s-\big(m^2+|k_{p+1}|^2\big)^s\big) \\ &\times\widetilde{\Pi}_{k_0}\Pi_{n_0}\,\mathfrak{a}_\ell(\tilde n;\Pi_{n_1}u_1,\dots,\Pi_{n_p}u_p)\Lambda_m^{-1}\Pi_{n_{p+1}}\widetilde{\Pi}_{k_{p+1}}u_{p+1}\end{aligned} \tag{6.42}$$

so that we have

$$\mathrm{V} = \sum_{\ell=0}^{p} Re\, i\big\langle M_\ell^{p,1}(\underbrace{\bar u,\dots,\bar u}_{\ell},\underbrace{u,\dots,u}_{p+1-\ell}),u\big\rangle, \tag{6.43}$$

where as before we have ruled out the indices $\vec{n} \in S_p^{\omega_\ell}$ in definition (6.42), so that (5.9) holds automatically with M replace by $M_\ell^{p,1}$ and for any $\vec{n} \in S_p^{\omega_\ell}$.

It follows by Lemma 4.5 and the cut-off function in the definition that item (i) of Definition 5.4 is satisfied for $M_\ell^{p,1}$. To establish (5.12) for $M_\ell^{p,1}$, we may assume that the left-hand side of (5.12) with M replaced by $M_\ell^{p,1}$ is nonzero. Thus we have for some $n_0, n_{p+1} \in \mathbb{N}$

$$\begin{aligned} |k_0 - k_{p+1}| &< \frac{1}{2}\big(|k_0| + |k_{p+1}|\big), \\ |k_0 - k_{p+1}| &\leqslant \lambda_{n'}, \\ \lambda_{n'} &< \delta\lambda_{n_{p+1}}, \\ |k_0| &= \lambda_{n_0}, \\ |k_{p+1}| &= \lambda_{n_{p+1}}, \end{aligned} \tag{6.44}$$

from which we also deduce $|k_0| \sim |k_{p+1}|$ and

$$\begin{aligned} \big|\big(m^2 + |k_0|^2\big)^s - \big(m^2 + |k_{p+1}|^2\big)^s\big| &\leqslant C\big(1 + |k_0 - k_{p+1}|\big)\big(1 + |k_0| + |k_{p+1}|\big)^{2s-1} \\ &\leqslant C(1 + \lambda_{n'})\big(1 + |k_0| + |k_{p+1}|\big)^{2s-1}. \end{aligned} \tag{6.45}$$

Then it follows by (iii) of Lemma 4.5 and Lemma 4.4 that (5.12) holds with M replaced by $M_\ell^{p,1}$ and for $\tau = 2s - 2$ and for some $\nu \in \mathbb{R}_+$. So we have $M_\ell^{p,1} \in \mathcal{G}_{p+1,\delta}^{\nu,2s-2}$. Since we have already seen that (5.9) holds, we then get by Proposition 5.5 that $M_\ell^{p,1} \in \mathcal{G}_{p+1,\delta}^{\nu,2s-2} \cap \widetilde{\mathcal{M}}_{p+1,\delta}^{\nu+d,2s-2}(\omega_\ell)$.

(iv) The term VI.

The analysis of the term VI is almost the same as that of V. We define

$$\begin{aligned} M_\ell^{p,2}(u_1, \ldots, u_{p+1}) = \frac{1}{2} \sum_{k_0, k_{p+1} \in \mathbb{Z}^d} \sum_{\vec{n} \in (S_p^{\omega_\ell})^c} &\mathbf{1}_{\{\lambda_{n'} < \delta\lambda_{n_{p+1}},\ |k_0 - k_{p+1}| < \frac{1}{2}(|k_0| + |k_{p+1}|)\}} \\ &\times \big(\big(m^2 + |k_{p+1}|^2\big)^{-1/2} - \big(m^2 + |k_0|^2\big)^{-1/2}\big) \\ &\times \widetilde{\Pi}_{k_0} \Pi_{n_0} \mathbf{a}_\ell(\tilde{n}; \Pi_{n_1} u_1, \ldots, \Pi_{n_p} u_p) \Lambda_m^{2s} \Pi_{n_{p+1}} \widetilde{\Pi}_{k_{p+1}} u_{p+1} \end{aligned} \tag{6.46}$$

so that we have

$$\mathrm{VI} = \sum_{\ell=0}^{p} Re\, i \big\langle M_\ell^{p,2}(\underbrace{\bar{u}, \ldots, \bar{u}}_{\ell}, \underbrace{u, \ldots, u}_{p+1-\ell}), u \big\rangle. \tag{6.47}$$

We assume that the left-hand side of (5.12) with M replaced by $M_\ell^{p,2}$ is nonzero, and we then have (6.44) and

$$\big|\big(m^2 + |k_{p+1}|^2\big)^{-1/2} - \big(m^2 + |k_0|^2\big)^{-1/2}\big| \leqslant C(1 + \lambda_{n'})\big(1 + |k_0| + |k_{p+1}|\big)^{-2}. \tag{6.48}$$

And again we get $M_\ell^{p,2} \in \mathcal{G}_{p+1,\delta}^{\nu,2s-2} \cap \widetilde{\mathcal{M}}_{p+1,\delta}^{\nu+d,2s-2}(\omega_\ell)$ for some $\nu \in \mathbb{R}_+$ and small δ.

Define M_ℓ^p, R_ℓ^p to be the operators

$$M_\ell^p(u_1, \ldots, u_{p+1}) = M_\ell^{p,1}(u_1, \ldots, u_{p+1}) + M_\ell^{p,2}(u_1, \ldots, u_{p+1}),$$

$$R_\ell^p(u_1, \dots, u_{p+1}) = R_\ell^{p,1}(u_1, \dots, u_{p+1}) + R_\ell^{p,2}(u_1, \dots, u_{p+1}).$$

This concludes the proof. □

Summarizing the above lemmas gives an end to the proof of Proposition 6.1. □

Remark 6.5. We will gain one derivative on the terms used to eliminate $\widetilde{M}_\ell^p$ which is of order $2s-1$, but no derivative on the terms used to eliminate M_ℓ^p.

7. Geometric bounds

Consider the function on $\mathbb{R}^{p+2}$ depending on the parameter $m \in (0, +\infty)$, defined for $\ell = 0, \dots, p+1$ by

$$F_m^\ell(\xi_0, \dots, \xi_{p+1}) = \sum_{j=0}^{\ell} \sqrt{m^2 + \xi_j^2} - \sum_{j=\ell+1}^{p+1} \sqrt{m^2 + \xi_j^2}. \tag{7.1}$$

The following result will play an important role in inverting the multilinear operators defined in Section 5.

Proposition 7.1. *For any $\rho > 0$, there is a zero measure subset $\mathcal{N}$ of $\mathbb{R}_+^*$ such that for any integers $0 \leqslant \ell \leqslant p+1$, any $m \in \mathbb{R}_+^* - \mathcal{N}$, there are constants $c > 0$, $N_0 \in \mathbb{N}$ such that the lower bound*

$$\left|F_m^\ell(\lambda_{n_0}, \dots, \lambda_{n_{p+1}})\right| \geqslant c(1 + \lambda_{n_0} + \lambda_{n_{p+1}})^{-3-\rho}\left(|\lambda_{n_0} - \lambda_{n_{p+1}}| + 1 + \lambda_{n'}\right)^{-N_0} \tag{7.2}$$

holds true for any $(n_0, \dots, n_{p+1}) \in \mathbb{N}^{p+2} - S_p^{\omega_\ell}$, where $(\lambda_n)_{n\in\mathbb{N}}$ is given by (3.4), $\lambda_{n'}$ by (3.6), and $S_p^{\omega_\ell}$ by (5.10) with $\omega_\ell(j) = -1$ for $j = 0, \dots, \ell$, and $\omega_\ell(j) = 1$ for $j = \ell+1, \dots, p+1$.

This proposition is an analogue of Theorem 2.3.1 in [10]. Let us assume that $\lambda_{n_0}, \lambda_{n_{p+1}} \gg \lambda_{n_1}, \dots, \lambda_{n_p}$ and briefly explain the way of the proof of (7.2). The reader may refer to [10] for more details. We first notice that one only needs to show that, for any compact interval $I \subset (0, +\infty)$, the measure of the set

$$\left\{m \in I;\ \left|F_m^\ell(\lambda_{n_0}, \dots, \lambda_{n_{p+1}})\right| < r\right\}$$

where r is the right-hand side of (7.2) with c replaced by α, goes to zero as α tends to zero. Using tools of subanalytic geometry, the interval I may be written for any fixed $n_0, \dots, n_{p+1}$ as the union of a uniform number of intervals on which $|\partial F_m^\ell/\partial m|$ can be bounded from below by a large negative power of small frequencies $(1 + \lambda_{n_1} + \dots + \lambda_{n_p})$, and of a remaining set. On each of these intervals, since we have $|\partial F_m^\ell/\partial m| \geqslant C(1 + \lambda_{n_1} + \dots + \lambda_{n_p})^{-N_1}$, we then take F_m^ℓ as a coordinate so that we can estimate the measure of this interval by $Cr(1 + \lambda_{n_1} + \dots + \lambda_{n_p})^{N_1}$. Taking the expression of r into account, we get an upper bound of the sum of these quantities in $n_0, \dots, n_{p+1}$ by a constant which goes to zero as α tends to zero. Also using tools of subanalytic geometry we can show that the measure of the remaining set, on which we have $|\partial F_m^\ell/\partial m| = O(1 + \lambda_{n_1} + \dots + \lambda_{n_p})^{-N_1}$, is small and goes to zero as α tends to zero. This shows that (7.2) holds true for all $(n_0, \dots, n_{p+1}) \in \mathbb{N}^{p+2} - S_p^{\omega_\ell}$ when m is outside a subset of zero measure in I.

For the difference between the estimate of type (7.2) in [5] and ours, one has to take into account the multiplicity of the eigenvalues of $\sqrt{-\Delta}$ on $\mathbb{T}^d$ in the framework of [5], while in our framework, the multiplicity of the eigenvalues does not play any role. Simply speaking, with the same reasoning as above, after getting an upper bound of the measure of the intervals on which $|\partial F_m^\ell/\partial m| \geqslant C(1 +$

$\lambda_{n_1} + \cdots + \lambda_{n_p})^{-N_1}$ and that of the remaining set, one has to take the sum over $n_0, \ldots, n_{p+1} \in \mathbb{Z}^d$ in the framework of [5], while in our method we only need to sum over $n_0, \ldots, n_{p+1} \in \mathbb{N}$. This makes the estimate different.

We shall need another proposition which is nothing but Proposition 2.1.5 in [5]. Before stating it, let us introduce some notations. For $m \in \mathbb{R}_+^*$, $\xi_j \in \mathbb{R}$, $j = 0, \ldots, p+1$, $e = (e_0, \ldots, e_{p+1}) \in \{-1, 1\}^{p+2}$, define

$$\widetilde{F}_m^{(e)}(\xi_0, \ldots, \xi_{p+1}) = \sum_{j=0}^{p+1} e_j \sqrt{m^2 + \xi_j^2}. \tag{7.3}$$

When p is even and $\#\{j;\ e_j = 1\} = p/2 + 1$, denote by $N^{(e)}$ the set of all $(n_0, \ldots, n_{p+1}) \in \mathbb{N}^{p+2}$ such that there is a bijection σ from $\{j;\ 0 \leqslant j \leqslant p+1,\ e_j = 1\}$ to $\{j;\ 0 \leqslant j \leqslant p+1,\ e_j = -1\}$ so that for any j in the first set $n_j = n_{\sigma(j)}$. In the other cases, set $N^{(e)} = \emptyset$.

Proposition 7.2. *There is a zero measure subset $\mathcal{N}$ of $\mathbb{R}_+^*$ and for any $m \in \mathbb{R}_+^* - \mathcal{N}$, there are constants $c > 0$, $N_0 \in \mathbb{N}$ such that for any $(n_0, \ldots, n_{p+1}) \in \mathbb{N}^{p+2} - N^{(e)}$ we have*

$$\left|\widetilde{F}_m^{(e)}(\lambda_{n_0}, \ldots, \lambda_{n_{p+1}})\right| \geqslant c(1 + \lambda_{n_0} + \cdots + \lambda_{n_{p+1}})^{-N_0}. \tag{7.4}$$

Moreover, if $e_0 e_{p+1} = 1$, we have the inequality

$$\left|\widetilde{F}_m^{(e)}(\lambda_{n_0}, \ldots, \lambda_{n_{p+1}})\right| \geqslant c(1 + \lambda_{n_0} + \lambda_{n_{p+1}})(1 + \lambda_{n_1} + \cdots + \lambda_{n_p})^{-N_0}. \tag{7.5}$$

8. Energy control and the proof of the main theorem

We shall show in this section that the H^s energy is finite on an interval so that the solution does not blow up on it. As we have pointed out in the introduction, we shall perturb the Sobolev energy in such a way that the time derivative of perturbations will cancel out the main contribution to that of the Sobolev energy, up to remainders of higher order. Moreover, the perturbations should be controlled properly. We first introduce some notations. When $M(u_1, \ldots, u_{p+1})$ is a $p+1$-linear form, let us define for $0 \leqslant \ell \leqslant p+1$,

$$\begin{aligned} L_\ell^-(M)(u_1, \ldots, u_{p+1}) = -\Lambda_m M(u_1, \ldots, u_{p+1}) - \sum_{j=1}^{\ell} M(u_1, \ldots, \Lambda_m u_j, \ldots, u_{p+1}) \\ + \sum_{j=\ell+1}^{p+1} M(u_1, \ldots, \Lambda_m u_j, \ldots, u_{p+1}) \end{aligned} \tag{8.1}$$

and

$$\begin{aligned} L_\ell^+(M)(u_1, \ldots, u_{p+1}) = -\Lambda_m M(u_1, \ldots, u_{p+1}) - M(u_1, \ldots, u_p, \Lambda_m u_{p+1}) \\ - \sum_{j=1}^{\ell} M(u_1, \ldots, \Lambda_m u_j, \ldots, u_{p+1}) \\ + \sum_{j=\ell+1}^{p} M(u_1, \ldots, \Lambda_m u_j, \ldots, u_{p+1}). \end{aligned} \tag{8.2}$$

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a cut-off function supported with small support.
We shall need the following lemma:

Lemma 8.1. *For any $\rho > 0$, let $\mathcal{N}$ be the zero measure subset of $\mathbb{R}_+^*$ defined by taking the union of the zero measure subsets defined in Propositions 7.1 and 7.2, and fix $m \in \mathbb{R}_+^* - \mathcal{N}$. Let ω_ℓ, $\widetilde{\omega}_\ell$ be defined as in the statement of Proposition 6.1. Then there is a $\bar{\nu} \in \mathbb{N}$ such that the following statements hold true for any large enough s, for any integer p with $\kappa \leqslant p \leqslant 2\kappa - 1$, for any integer ℓ with $0 \leqslant \ell \leqslant p$:*

(i) *Let $\theta \in (0,1)$, $M_\ell^p \in \mathcal{G}_{p+1,\delta}^{\nu,2s-2} \cap \widetilde{\mathcal{M}}_{p+1,\delta}^{\nu+d,2s-2}(\omega_\ell)$ and $\widetilde{M}_\ell^p \in \widetilde{\mathcal{M}}_{p+1,\delta}^{\nu,2s-1}(\widetilde{\omega}_\ell)$. Define*

$$M_\ell^{p,\epsilon}(u_1,\dots,u_{p+1}) = M_\ell^p\big(\chi\big(\epsilon^{\kappa\theta}\Lambda_m\big)u_1,\dots,\chi\big(\epsilon^{\kappa\theta}\Lambda_m\big)u_{p+1}\big). \tag{8.3}$$

Then there are $\underline{M}_\ell^{p,\epsilon} \in \widetilde{\mathcal{M}}_{p+1,\delta}^{\nu+\bar{\nu},2s-1}(\omega_\ell)$ and $\underline{\widetilde{M}}_\ell^p \in \widetilde{\mathcal{M}}_{p+1,\delta}^{\nu+\bar{\nu},2s-2}(\widetilde{\omega}_\ell)$ satisfying

$$\begin{aligned} L_\ell^-\big(\underline{M}_\ell^{p,\epsilon}\big)(u_1,\dots,u_{p+1}) &= M_\ell^{p,\epsilon}(u_1,\dots,u_{p+1}),\\ L_\ell^+\big(\underline{\widetilde{M}}_\ell^p\big)(u_1,\dots,u_{p+1}) &= \widetilde{M}_\ell^p(u_1,\dots,u_{p+1}) \end{aligned} \tag{8.4}$$

with the estimate,

$$\begin{aligned} \big\|\underline{M}_\ell^{p,\epsilon}\big\|_{\mathcal{M}_{p+1,\delta}^{\nu+\bar{\nu},2s-1}} &\leqslant C\epsilon^{-(2+\rho)\theta\kappa}\big\|M_\ell^p\big\|_{\mathcal{M}_{p+1,\delta}^{\nu,2s-2}},\\ \big\|\underline{\widetilde{M}}_\ell^p\big\|_{\mathcal{M}_{p+1,\delta}^{\nu+\bar{\nu},2s-2}} &\leqslant C\big\|\widetilde{M}_\ell^p\big\|_{\mathcal{M}_{p+1,\delta}^{\nu,2s-1}}, \end{aligned} \tag{8.5}$$

where $\|\cdot\|_{\mathcal{M}_{p+1,\delta}^{\nu,\tau}}$ is defined in the statement of Definition 5.1.

(ii) *Let $R_\ell^p \in \widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\omega_\ell)$, $\widetilde{R}_\ell^p \in \widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\widetilde{\omega}_\ell)$. Then there are $\underline{R}_\ell^p \in \widetilde{\mathcal{R}}_{p+1}^{\nu+\bar{\nu},2s}(\omega_\ell)$ and $\underline{\widetilde{R}}_\ell^p \in \widetilde{\mathcal{R}}_{p+1}^{\nu+\bar{\nu},2s}(\widetilde{\omega}_\ell)$ such that*

$$\begin{aligned} L_\ell^-\big(\underline{R}_\ell^p\big)(u_1,\dots,u_{p+1}) &= R_\ell^p(u_1,\dots,u_{p+1}),\\ L_\ell^+\big(\underline{\widetilde{R}}_\ell^p\big)(u_1,\dots,u_{p+1}) &= \widetilde{R}_\ell^p(u_1,\dots,u_{p+1}). \end{aligned} \tag{8.6}$$

Proof. (i) First we have $M_\ell^{p,\epsilon} \in \widetilde{\mathcal{M}}_{p+1,\delta}^{\nu+d,2s-2}(\omega_\ell)$ by assumption. We then substitute in (8.4) $\Pi_{n_j}u_j$ for u_j, $j = 1,\dots,p+1$, and compose on the left with Π_{n_0}. According to (8.1), equalities in (8.4) may be written as

$$\begin{aligned} &-F_m^\ell(\lambda_{n_0},\dots,\lambda_{n_{p+1}})\Pi_{n_0}\underline{M}_\ell^{p,\epsilon}(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})\\ &\quad= \Pi_{n_0}M_\ell^{p,\epsilon}(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1}),\\ &\widetilde{F}_m^{(e)}(\lambda_{n_0},\dots,\lambda_{n_{p+1}})\Pi_{n_0}\underline{\widetilde{M}}_\ell^p(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})\\ &\quad= \Pi_{n_0}\widetilde{M}_\ell^p(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1}), \end{aligned} \tag{8.7}$$

where F_m^ℓ is defined by (7.1) and $\widetilde{F}_m^{(e)}$ is defined by (7.3) with

$$e_0 = \dots = e_\ell = e_{p+1} = -1, \qquad e_{\ell+1} = \dots = e_p = 1.$$

When considering the first equality of (8.7), we assume its right-hand side does not vanish. By the definition this implies

$$|\lambda_{n_0} - \lambda_{n_{p+1}}| \leqslant \lambda_{n'}, \quad (n_0, \dots, n_{p+1}) \notin S_p^{\omega_\ell} \tag{8.8}$$

with $S_p^{\omega_\ell}$ the same as that in Proposition 7.1. Therefore the assumptions of Proposition 7.1 are satisfied. Thus for any $\rho > 0$, there is a zero measure subset $\mathcal{N}$ of $\mathbb{R}_+$ such that we have by (7.2), (8.8) and the condition $\lambda_{n_0} + \lambda_{n_{p+1}} < \epsilon^{-\theta\kappa}$

$$\begin{aligned}\left|F_m^\ell(\lambda_{n_0}, \dots, \lambda_{n_{p+1}})\right|^{-1} &\leqslant C(1 + \lambda_{n_0} + \lambda_{n_{p+1}})^{3+\rho}\left(|\lambda_{n_0} - \lambda_{n_{p+1}}| + 1 + \lambda_{n'}\right)^{N_0} \\ &\leqslant C\epsilon^{-(2+\rho)\theta\kappa}(1 + \lambda_{n_0} + \lambda_{n_{p+1}})(1 + \lambda_{n'})^{N_0}\end{aligned} \tag{8.9}$$

for some $N_0 > 0$. Consequently, if we set

$$\underline{M}_\ell^{p,\epsilon}(u_1, \dots, u_{p+1}) = -\sum_{\vec{n} \notin S_p^\ell} F_m^\ell(\lambda_{n_0}, \dots, \lambda_{n_{p+1}})^{-1} \Pi_{n_0} M_\ell^{p,\epsilon}(\Pi_{n_1} u_1, \dots, \Pi_{n_{p+1}} u_{p+1}), \tag{8.10}$$

we obtain according to (8.9) and (5.2) that $\underline{M}_\ell^{p,\epsilon} \in \widetilde{\mathcal{M}}_{p+1,\delta}^{\nu+\bar{\nu},2s-1}(\omega_\ell)$ with the first estimate in (8.5) with $\bar{\nu} = N_0 + d$.

When considering the second equality of (8.7), we also assume its right-hand side does not vanish. Therefore, we know from (5.9) with M replaced by $\widetilde{M}_\ell^p$ that there is no bijection σ from $\{0, \dots, \ell, p+1\}$ to $\{\ell+1, \dots, p\}$ such that $n_j = n_{\sigma(j)}$, $j = 0, \dots, \ell, p+1$. Consequently, the condition of Proposition 7.2 is satisfied and we may use (7.5). We divide in both sides by $\widetilde{F}_m^{(e)}$ and add up all these nonzero components as in (8.10) to get an element $\underline{\widetilde{M}}_\ell^p$, which is easily seen to be in the class $\widetilde{\mathcal{M}}_{p+1,\delta}^{\nu+\bar{\nu},2s-2}(\widetilde{\omega}_\ell)$ for some $\bar{\nu}$. This completes the proof (i) of Lemma 8.1.

(ii) We deduce again from (8.6)

$$-F_m^\ell(\lambda_{n_0}, \dots, \lambda_{n_{p+1}})\Pi_{n_0} \underline{R}_\ell^p(\Pi_{n_1} u_1, \dots, \Pi_{n_{p+1}} u_{p+1}) = \Pi_{n_0} R_\ell^p(\Pi_{n_1} u_1, \dots, \Pi_{n_{p+1}} u_{p+1}), \tag{8.11}$$

$$\widetilde{F}_m^{(e)}(\lambda_{n_0}, \dots, \lambda_{n_{p+1}})\Pi_{n_0} \underline{\widetilde{R}}_\ell^p(\Pi_{n_1} u_1, \dots, \Pi_{n_{p+1}} u_{p+1}) = \Pi_{n_0} \widetilde{R}_\ell^p(\Pi_{n_1} u_1, \dots, \Pi_{n_{p+1}} u_{p+1}), \tag{8.12}$$

where F_m^ℓ and $\widetilde{F}_m^{(e)}$ are the same as in (8.7). Since $R_\ell^p \in \widetilde{\mathcal{R}}_{p+1}^{\nu,2s}(\omega_\ell)$, we have $(n_0, \dots, n_{p+1}) \notin S_p^{\omega_\ell}$ if the right-hand side of (8.11) does not vanish. This allows us to use Proposition 7.1 to get for some $N_0 > 0$

$$\left|F_m^\ell(\lambda_{n_0}, \dots, \lambda_{n_{p+1}})\right|^{-1} \leqslant C(1 + \lambda_{n_0} + \lambda_{n_1} + \dots + \lambda_{n_{p+1}})^{N_0+4}.$$

Dividing (8.11) by $-F_m^\ell$, we define as in (8.10) an element $\underline{R}_\ell^p$ in $\widetilde{\mathcal{R}}_{p+1}^{\nu+\bar{\nu},2s}(\omega_\ell)$ with $\bar{\nu} = N_0 + 4$. With the same reasoning we may use the inequality (7.4) in Proposition 7.2 with $e_0 = \dots = e_\ell = e_{p+1} = -1$, $e_{\ell+1} = \dots = e_p = 1$ and get from (8.12) an element $\underline{\widetilde{R}}_\ell^p \in \widetilde{\mathcal{R}}_{p+1}^{\nu+\bar{\nu},2s}(\widetilde{\omega}_\ell)$ for some $\bar{\nu}$. This concludes the proof. □

Proposition 8.2. *For any $\rho > 0$, let $\mathcal{N}$ be the zero measure subset of $\mathbb{R}_+^*$ defined in Lemma 8.1, and fix $m \in \mathbb{R}_+^* - \mathcal{N}$. Let Θ_s be defined in (6.5). Then there are for any large enough s, a map Θ_s^1, sending $H^s(\mathbb{T}^d) \times (0, 1/2)$ to $\mathbb{R}$, and maps Θ_s^2, Θ_s^3, Θ_s^4 sending $H^s(\mathbb{T}^d)$ to $\mathbb{R}$ such that there is a constant $C_s > 0$ and for any $u \in H^s(\mathbb{T}^d)$ with $\|u\|_{H^s} \leqslant 1$ and any $\epsilon \in (0, 1/2)$, we have*

$$\left|\Theta_s^1(u, \epsilon)\right| \leqslant C_s \epsilon^{-(2+\rho)\theta\kappa} \|u\|_{H^s}^{\kappa+2}, \qquad \left|\Theta_s^j(u)\right| \leqslant C_s \|u\|_{H^s}^{\kappa+2}, \quad j = 2, 3, 4, \tag{8.13}$$

and such that

$$R(u) \stackrel{def}{=} \frac{d}{dt}\left[\Theta_s\big(u(t,\cdot)\big) - \Theta_s^1\big(u(t,\cdot), \epsilon\big) - \Theta_s^2\big(u(t,\cdot)\big) - \Theta_s^3\big(u(t,\cdot)\big) - \Theta_s^4\big(u(t,\cdot)\big)\right] \tag{8.14}$$

satisfies

$$\left|R(u)\right| \leqslant C_s\epsilon^{-(2+\rho)\theta\kappa}\|u\|_{H^s}^{2\kappa+2} + C_s\epsilon^{2\theta\kappa}\|u\|_{H^s}^{\kappa+2} + C_s\|u\|_{H^s}^{2\kappa+2}. \tag{8.15}$$

Proof. Considering the right-hand side of (6.6), we decompose

$$M_\ell^p(u_1,\dots,u_{p+1}) = M_\ell^{p,\epsilon}(u_1,\dots,u_{p+1}) + V_\ell^{p,\epsilon}(u_1,\dots,u_{p+1}), \tag{8.16}$$

where the first term in the right-hand side is given by (8.3) and the second one is the sum of terms like

$$M_\ell^p\big(\chi_1\big(\epsilon^{\kappa\theta}\Lambda_m\big)u_1,\dots,\chi_{p+1}\big(\epsilon^{\kappa\theta}\Lambda_m\big)u_{p+1}\big), \tag{8.17}$$

where $\chi_j = \chi$ or $1-\chi$, $j=1,\dots,p+1$, and where there is at least one $j_0 \in \{1,\dots,p+1\}$ such that $\chi_{j_0} = 1-\chi$.

We claim that the H^{-s} norm of $V_\ell^{p,\epsilon}(u_1,\dots,u_{p+1})$ is controlled by $C\epsilon^{2\kappa\theta}\prod_{j=0}^{p+1}\|u_j\|_{H^s}$ for large enough s. To see this, we first note that

$$(u_1,\dots,u_{p+1}) \to M_\ell^p\big(\chi_1\big(\epsilon^{\kappa\theta}\Lambda_m\big)u_1,\dots,\chi_{p+1}\big(\epsilon^{\kappa\theta}\Lambda_m\big)u_{p+1}\big)$$

defines an element of $\mathcal{G}_{p+1,\delta}^{\nu,2s-2}$ since $M_\ell^p \in \mathcal{G}_{p+1,\delta}^{\nu,2s-2}$. Then if $j_0 \in \{1,\dots,p\}$ is such that $\chi_{j_0} = 1-\chi$, without loss of generality, we take $j_0 = 1$, i.e., $\chi_1 = 1-\chi$. Then by Proposition 5.6 with s replaced by $s-2$ and $\tau = 2s-2$ we bound the H^{-s} norm of (8.17) from above by a constant times

$$\big\|(1-\chi)\big(\epsilon^{\kappa\theta}\Lambda_m\big)u_1\big\|_{H^{s_0}}\prod_{j=2}^{p}\|u_j\|_{H^{s_0}}\|u_{p+1}\|_{H^{s-2}}. \tag{8.18}$$

But

$$\big\|(1-\chi)\big(\epsilon^{\kappa\theta}\Lambda_m\big)u_1\big\|_{H^{s_0}} \leqslant C\epsilon^{(s-s_0)\kappa\theta}\|u_1\|_{H^s} \leqslant C\epsilon^{2\kappa\theta}\|u_1\|_{H^s}$$

if $s > s_0 + 2$. Then we get an upper bound of H^{-s} norm of (8.17) by $C\epsilon^{2\kappa\theta}\prod_{j=1}^{p+1}\|u_j\|_{H^s}$. Now if $j_0 = p+1$ is such that $\chi_{j_0} = 1-\chi$, i.e., $\chi_{p+1} = 1-\chi$. Again, by Proposition 5.6 with s replaced by $s-2$ and $\tau = 2s-2$, the H^{-s} norm of (8.17) is bounded from above by a constant times

$$\prod_{j=1}^{p}\|u_j\|_{H^{s_0}}\big\|(1-\chi)\big(\epsilon^{\kappa\theta}\Lambda_m\big)u_{p+1}\big\|_{H^{s-2}},$$

which turns out to be controlled by $C\epsilon^{2\kappa\theta}\prod_{j=1}^{p+1}\|u_j\|_{H^s}$. This proves our claim.

Consequently, the quantity

$$\sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\, i\big\langle V_\ell^{p,\epsilon}(\bar u,\dots,\bar u,u,\dots,u),u\big\rangle \tag{8.19}$$

is bounded form above by the second term of the right-hand side of (8.15) if $\|u\|_{H^s} \leqslant 1$. In the rest of the proof, we may therefore replace in the right-hand side of (6.6) M_ℓ^p by $M_\ell^{p,\epsilon}$.

Applying Lemma 8.1 to $M_\ell^{p,\epsilon}$, $\widetilde{M}_\ell^p$, R_ℓ^p, $\widetilde{R}_\ell^p$ gives $\underline{M}_\ell^{p,\epsilon}$, $\underline{\widetilde{M}}_\ell^p$, $\underline{R}_\ell^p$, $\underline{\widetilde{R}}_\ell^p$. We set

$$\begin{aligned}
\Theta_s^1\big(u(t,\cdot),\epsilon\big) &= \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\big\langle \underline{M}_\ell^{p,\epsilon}(\bar u,\dots,\bar u,u,\dots,u),u\big\rangle,\\
\Theta_s^2\big(u(t,\cdot)\big) &= \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\big\langle \underline{\widetilde{M}}_\ell^{p}(\bar u,\dots,\bar u,u,\dots,u,\bar u),u\big\rangle,\\
\Theta_s^3\big(u(t,\cdot)\big) &= \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\big\langle \underline{R}_\ell^{p}(\bar u,\dots,\bar u,u,\dots,u),u\big\rangle,\\
\Theta_s^4\big(u(t,\cdot)\big) &= \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\big\langle \underline{\widetilde{R}}_\ell^{p}(\bar u,\dots,\bar u,u,\dots,u,\bar u),u\big\rangle.
\end{aligned} \tag{8.20}$$

The general term in $\Theta_s^1(u(t,\cdot),\epsilon)$ has modulus bounded from above by

$$\big\|\underline{M}_\ell^{p,\epsilon}(\bar u,\dots,\bar u,u,\dots,u)\big\|_{H^{-s}}\|u\|_{H^s} \leqslant C\epsilon^{-(2+\rho)\theta\kappa}\|u\|_{H^s}^{\kappa+2}$$

for u in the unit ball of $H^s(\mathbb{T}^d)$, using Proposition 5.2 with $\tau = 2s-1$ and (8.5) in Lemma 8.1. This gives the first inequality in (8.13). We apply Proposition 5.2 to $\underline{\widetilde{M}}_\ell^p$, remarking that if in (5.3) $\tau = 2s-1$ and s is large enough, the left-hand side of (5.3) controls the H^{-s} norm of $\underline{\widetilde{M}}_\ell^p(\bar u,\dots,\bar u,u,\dots,u,\bar u)$. We also apply Proposition 5.8 with $\tau = 2s$ in (5.23) to $\underline{R}_\ell^p$, $\underline{\widetilde{R}}_\ell^p$. Then if s_0 is large enough, the left-hand side of (5.23) controls H^{-s} norm of $\underline{R}_\ell^p(\bar u,\dots,\bar u,u,\dots,u)$ and $\underline{\widetilde{R}}_\ell^p(\bar u,\dots,\bar u,u,\dots,u,\bar u)$. These give us the other inequalities in (8.13). Consequently we are left with proving (8.15). Remarking that we may also write the equation as

$$(D_t - \Lambda_m)u = -F\left(\Lambda_m^{-1}\left(\frac{u+\bar u}{2}\right)\right), \tag{8.21}$$

we compute using notation (8.1)

$$\begin{aligned}
\frac{d}{dt}\Theta_s^1(u,\epsilon) = {}& \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\, i\big\langle L_\ell^-\big(\underline{M}_\ell^{p,\epsilon}\big)(\bar u,\dots,\bar u,u,\dots,u),u\big\rangle\\
&+ \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p}\sum_{j=1}^{\ell} Re\, i\big\langle \underline{M}_\ell^{p,\epsilon}(\bar u,\dots,\bar F,\dots,\bar u,u,\dots,u),u\big\rangle\\
&- \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p}\sum_{j=\ell+1}^{p+1} Re\, i\big\langle \underline{M}_\ell^{p,\epsilon}(\bar u,\dots,\bar u,u,\dots,F,\dots,u),u\big\rangle\\
&+ \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\, i\big\langle \underline{M}_\ell^{p,\epsilon}(\bar u,\dots,\bar u,u,\dots,u),F\big\rangle.
\end{aligned} \tag{8.22}$$

By assumption on F, we have by Lemma 4.2 $\|F(v)\|_{H^s} \leqslant C\|u\|_{H^s}^{\kappa+1}$ if s is large enough and $\|u\|_{H^s} \leqslant 1$. Since $\underline{M}_\ell^{p,\epsilon} \in \widetilde{\mathcal{M}}_{p+1}^{\nu+\bar\nu,2s-1}(\omega_\ell)$, we may apply Proposition 5.2 with $\tau = 2s-1$ and (8.5) to see that the last three terms in (8.22) have modulus bounded from above by the first term in the right-hand side of (8.15). When computing $\frac{d}{dt}\Theta_s(u)$, noting that we have replaced M_ℓ^p by $M_\ell^{p,\epsilon}$, the first term in the right-hand side of (6.6) is the first term in the right-hand side of (8.22) because of (8.4). Consequently, these contributions will cancel out each other in the expression $\frac{d}{dt}[\Theta_s(u) - \Theta_s^1(u,\epsilon)]$. We compute

$$\begin{aligned}
\frac{d}{dt}\Theta_s^2(u) &= \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\, i\big\langle L_\ell^+\big(\underline{\widetilde{M}}_\ell^p\big)(\bar u,\dots,\bar u,u,\dots,u,\bar u),u\big\rangle \\
&+ \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p}\sum_{j=1}^{\ell} Re\, i\big\langle \underline{\widetilde{M}}_\ell^p(\bar u,\dots,\bar F,\dots,\bar u,u,\dots,u,\bar u),u\big\rangle \\
&- \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p}\sum_{j=\ell+1}^{p} Re\, i\big\langle \underline{\widetilde{M}}_\ell^p(\bar u,\dots,\bar u,u,\dots,F,\dots,u,\bar u),u\big\rangle \\
&+ \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\, i\big\langle \underline{\widetilde{M}}_\ell^p(\bar u,,\dots,\bar u,u,\dots,u,\bar F),u\big\rangle \\
&+ \sum_{p=\kappa}^{2\kappa-1}\sum_{\ell=0}^{p} Re\, i\big\langle \underline{\widetilde{M}}_\ell^p(\bar u,\dots,\bar u,u,\dots,u,\bar u),F\big\rangle.
\end{aligned} \tag{8.23}$$

Since $\underline{\widetilde{M}}_\ell^p \in \widetilde{\mathcal{M}}_{p+1}^{\nu+\bar\nu,2s-2}(\widetilde{\omega}_\ell)$, we have by Proposition 5.2 with $\tau = 2s-1$ and (8.5) that the last four terms in the right-hand side of (8.23) are estimated by the last term in the right-hand side of (8.15) if s is large enough. The first one, according to Lemma 8.1, cancels the contribution of $\widetilde{M}_\ell^p$ in (6.6) when computing $R(u)$. We may treat $\Theta_s^3(u)$ and $\Theta_s^4(u)$ in the same way using Proposition 5.8 with $\tau = 2s$, and this will lead to the third term in the right-hand side of (8.15). Finally, the last term in (6.6) contributes to the last term in the right-hand side of (8.15). This concludes the proof of the proposition. □

Proof of Theorem 2.1. We deduce from (8.14) and (8.15)

$$\begin{aligned}
\Theta_s\big(u(t,\cdot)\big) &\leqslant \Theta_s\big(u(0,\cdot)\big) - \Theta_s^1\big(u(0,\cdot),\epsilon\big) - \Theta_s^2\big(u(0,\cdot)\big) - \Theta_s^3\big(u(0,\cdot)\big) - \Theta_s^4\big(u(0,\cdot)\big) \\
&+ \Theta_s^1\big(u(t,\cdot),\epsilon\big) + \Theta_s^2\big(u(t,\cdot)\big) + \Theta_s^3\big(u(t,\cdot)\big) + \Theta_s^4\big(u(t,\cdot)\big) \\
&+ C_s\epsilon^{-(2+\rho)\theta\kappa}\int_0^t \big\|u\big(t',\cdot\big)\big\|_{H^s}^{2\kappa+2}\,dt' + C_s\epsilon^{2\theta\kappa}\int_0^t \big\|u\big(t',\cdot\big)\big\|_{H^s}^{\kappa+2}\,dt' \\
&+ C_s\int_0^t \big\|u\big(t',\cdot\big)\big\|_{H^s}^{2\kappa+2}\,dt'.
\end{aligned} \tag{8.24}$$

Take $\theta = 1/(4+\rho)$ and $B > 1$ a constant such that for any (v_0, v_1) in the unit ball of $H^{s+1}(\mathbb{T}^d) \times H^s(\mathbb{T}^d)$, $u(0,\cdot) = \epsilon(-iv_1 + \Lambda_m v_0)$ satisfies $\|u(0,\cdot)\|_{H^s} \leqslant B\epsilon$. Let $K > B$ be another constant to be chosen, and assume that for τ' in some interval $[0,T]$ we have $\|u(\tau',\cdot)\|_{H^s} \leqslant K\epsilon \leqslant 1$. From (8.13) and (8.24) we deduce that there is a constant $C > 0$, independent of B, K, ϵ, such that as long as $t \in [0,T]$

$$\big\|u(t,\cdot)\big\|_{H^s}^2 \leqslant C\big[B^2 + \epsilon^{\frac{2}{4+\rho}\kappa}K^{\kappa+2} + t\epsilon^{\frac{6+\rho}{4+\rho}\kappa}\big(K^{2\kappa+2} + K^{\kappa+2}\big) + t\epsilon^{2\kappa}K^{2\kappa+2}\big]\epsilon^2.$$

If we assume that $T \leqslant c\epsilon^{-\frac{6+\rho}{4+\rho}\kappa}$, where $\rho > 0$ is arbitrary and fixed in advance, for a small enough $c > 0$, and that ϵ is small enough, we get $\|u(t,\cdot)\|_{H^s}^2 \leqslant C(2B^2)\epsilon^2$. If K has been chosen initially so that $2CB^2 < K^2$, we get by a standard continuity argument that the priori bound $\|u(t,\cdot)\|_{H^s} \leqslant K\epsilon$ holds true on $[-c\epsilon^{-\frac{6+\rho}{4+\rho}\kappa}, c\epsilon^{-\frac{6+\rho}{4+\rho}\kappa}]$, in other words, the solution at least extends to such an interval $|t| \leqslant c\epsilon^{-\frac{6+\rho}{4+\rho}\kappa}$. Note that $c\epsilon^{-\frac{6+\rho}{4+\rho}\kappa} > c\epsilon^{-(3/2-\rho)\kappa}$ if ϵ is small and $\rho > 0$. This concludes the proof of the theorem. □

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Chapter 2

Growth of Sobolev Norms

On growth of Sobolev norms in linear Schrödinger equations with time dependent Gevrey potential

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Abstract

We improve Delort's method to show that solutions of linear Schrödinger equations with a time dependent Gevrey potential on the torus, have at most logarithmically growing Sobolev norms. In particular, it contains the result of Wang [7], which deals with analytic potentials in dimension 1.

Keywords: Sobolev norms ; Time dependent Schrödinger equation ; Gevrey potential

1 Introduction and statement of the theorem

The main goal of this paper is to obtain logarithmic growth of Sobolev norms of solutions of linear Schrödinger equations with a time dependent Gevrey potential on the torus, using the method of Delort [4]. Let $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and let $\mathbb{T}^d$ denote the standard torus, where $d \in \mathbb{N}^*$. We consider the time dependent linear Schrödinger equations:

$$i\partial_t u - \Delta u + V(x,t)u = 0 \tag{1.1}$$

on $\mathbb{T}^d \times \mathbb{R}$. We assume that the potential V is a real smooth function on $\mathbb{T}^d \times \mathbb{R}$. Let $\mu, \lambda \in [1, +\infty)$. We further assume that V is a Gevrey-μ function in time t and Gevrey-λ in every space variable, i.e., $V(x,t)$ satisfies estimates

$$\sup_{t\in\mathbb{R}} \sup_{x\in\mathbb{T}^d} |\partial_t^k \partial_x^\alpha V(x,t)| \leq C^{k+|\alpha|+1}(k!)^\mu(\alpha!)^\lambda \tag{1.2}$$

for any $k \in \mathbb{N}$, for any $\alpha \in \mathbb{N}^d$ and for some constant C independent of k and α.

We prove the following result:

Theorem 1. *There exists $\zeta > 0$ independent of μ and λ such that for any $s > 0$, there is a constant $C_{s,\lambda,d} > 0$ such that*

$$\|u(t)\|_{H^s} \leq C_{s,\lambda,d}\big[\log(2+|t|)\big]^{\zeta\mu\lambda s}\|u(0)\|_{H^s}, \tag{1.3}$$

where $u(t)$ is the solution to (1.1) with the initial condition $u_0 \overset{def}{=} u(0) \in H^s(\mathbb{T}^d)$.

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Remark 1.1. *Wang [7] obtained* (1.3) *with the exponent '$\zeta\mu\lambda s$' replaced by 'ζs' and $C_{s,\lambda,d}$ replaced by C_s under the assumption that the dimension $d = 1$ and that the potential $V(x,t)$ is bounded and analytic in space and time on $\Omega_{\tilde{\rho}}$ ($\tilde{\rho} > 0$ is a constant) when V is identified with a periodic function on $\mathbb{R}^d \times \mathbb{R}$, where*

$$\Omega_{\tilde{\rho}} = \{(x,t) \in \mathbb{C} \times \mathbb{C} \ : \ |\, Im\, x\,| < \tilde{\rho}, \ |\, Im\, t\,| < \tilde{\rho}\}.$$

When $d = 1$, the assumption we made here on the potential V is weaker than the assumption that V is analytic both in space and time on the strip $\Omega_{\tilde{\rho}}$, since the latter implies that V is a function of Gevrey-1 in time and Gevrey-1 in space. Moreover, our result concerns the case of any dimension $d \in \mathbb{N}^*$ instead of just $d = 1$.

Remark 1.2. *One may assume that $V(x,t)$ is a Gevrey-μ function in time and Gevrey-λ_i in space variable x_i for $1 \le i \le d$ with $\mu, \lambda_i \in [1, +\infty)$. However, this leads to* (1.2) *if we take $\lambda = \max\{\lambda_i \ : \ 1 \le i \le d\}$, and thus we may obtain the same result.*

The problem of finding optimal bounds for $\|u(t,\cdot)\|_{H^s}$ has been addressed by Nenciu [6] and Barbaroux et al. [1], in the abstract framework of an operator P (instead of $-\Delta$) and a perturbation $V(t)$ acting on elements of a Hilbert space, when the spectrum of P is discrete and has increasing gaps. This condition is satisfied by the Laplacian on the circle. It follows from the results of [6, 1], that solutions of (1.1) verify

$$\|u(t,\cdot)\|_{H^s} \le C_\epsilon |t|^\epsilon \|u(0,\cdot)\|_{H^s} \tag{1.4}$$

when t goes to infinity, for any $\epsilon > 0$. Later, Bourgain [3] proved that a similar bound holds for solutions of (1.1) on the torus $\mathbb{T}^d$. The increasing gap condition of Nenciu [6] and Barbaroux et al. [1] is no longer satisfied, and has to be replaced by a convenient decomposition of $\mathbb{Z}^d$ in well separated clusters. Delort [4] recently published a simpler proof of the results of Bourgain (included for other examples of compact manifolds than the torus), which is close to the one of Nenciu and Barbaroux-Joye. If one further assumes that V is analytic, and quasi-periodic in t, then it was showed by Bourgain [2] that (1.4) holds with $(1 + |t|)^\epsilon$ replaced by some power of $\log t$ when $t > 2$. When the dimension $d = 1$, for any real analytic potential, whose holomorphic extension to $\Omega_{\tilde{\rho}}$ is bounded, Wang [7] showed that one may still obtain such a logarithmic bound, using the method of [3]. In this paper, we improve the method of Delort [4] to provide a new proof of the result of Wang [7] and extend it to any dimension $d \ge 1$ and to Gevrey regularity.

There are also some results about uniformly bounded Sobolev norms. Eliasson et al. [5] have shown that if the potential V on $\mathbb{T}^d \times \mathbb{R}$ is analytic in space, quasi-periodic in time, and small enough, then for most values of the parameter of quasi-periodicity, the equation reduces to an autonomous one. Consequently, the Sobolev norm of the solution is uniformly bounded. A similar result for the harmonic oscillator has been obtained by Grébert and Thomann recently. For Schrödinger equations on the circle with a small time periodic potential, Wang [8] showed that the solutions of the corresponding equation have bounded Sobolev norms.

Now let us give a picture of the proof of Theorem 1. For any given $N \in \mathbb{N}^*$, one first finds for every fixed time t an operator $Q^N(\cdot,t)$, which extends as a bounded linear operator from $H^N(\mathbb{T}^d)$ to $H^N(\mathbb{T}^d)$ such that

$$(I + Q^N(\cdot,t))^*(i\partial_t - \Delta + V)(I + Q^N(\cdot,t)) = i\partial_t - \Delta + V_N'(\cdot,t) + R_N'(\cdot,t) \tag{1.5}$$

with self-adjoint operator V_N' exactly commuting to the modified Laplacian $\tilde{\Delta}$ (see (2.4) for its precise definition) and R_N' a remainder operator which is essentially a bounded linear map from $L^2(\mathbb{T}^d)$ to $H^N(\mathbb{T}^d)$. Moreover, we also require that the adjoint of Q^N in the usual L^2 paring (denoted by $Q^N(\cdot,t)^*$) extends as a bounded linear operator from $H^N(\mathbb{T}^d)$ to $H^N(\mathbb{T}^d)$. In order to obtain the estimate for the solution u of (1.1), one needs to 'invert' the operator $I + Q^N$, that is , to find an operator P^N, which extends as a bounded linear operator not only from $H^N(\mathbb{T}^d)$ to $H^N(\mathbb{T}^d)$, but also from $L^2(\mathbb{T}^d)$ to $L^2(\mathbb{T}^d)$, such that

$$(I + Q^N(\cdot,t))(I + P^N(\cdot,t)) = I + R_N(\cdot,t) \tag{1.6}$$

where R_N is a remainder operator such that $[i\partial_t - \Delta + V, R_N]$ sends $L^2(\mathbb{T}^d)$ to $H^N(\mathbb{T}^d)$. Now by setting

$$v = (I + P^N)u, \tag{1.7}$$

we deduce from (1.5), (1.6) and (1.1)

$$(i\partial_t - \Delta + V'_N)v = (I + Q^N)^*[i\partial_t - \Delta + V, R_N]u - R'_N v. \tag{1.8}$$

Remarking that the modified Laplacian has the property that

$$C^{-N}\|(1-\Delta)^{\frac{N}{2}}u\|_{L^2} \le \|(1-\tilde{\Delta})^{\frac{N}{2}}u\|_{L^2} \le C^N\|(1-\Delta)^{\frac{N}{2}}u\|_{L^2}$$

holds for any $u \in H^N(\mathbb{T}^d)$ and for some uniform constant C, then we let the operator $(1-\tilde{\Delta})^{\frac{N}{2}}$ act on both sides of (1.8) and deduce from the energy inequality

$$\|v(t)\|_{H^N} \le C_N\|v(0)\|_{H^N} + C_N\int_0^t \|(I+Q^N)^*[i\partial_t - \Delta + V, R_N]u(t)\|_{H^N} + \|R'_N v(t)\|_{H^N} dt,$$

which together with (1.7), the conservation law of the L^2-norm of (1.1) and the properties of those operators we have constructed, implies

$$\|v(t)\|_{H^N} \le C_N\|v(0)\|_{H^N} + C_N|t|\|u(0)\|_{L^2}. \tag{1.9}$$

We then use (1.6), (1.7) and the properties of the operators to deduce

$$\|u(t)\|_{H^N} \le C_N\Big(\|u(0)\|_{H^N} + (2+|t|)\|u(0)\|_{L^2}\Big). \tag{1.10}$$

Remark that the above constants C_N may be different in different lines and they depend on the norms of operators which appear in the above process. Since (1.10) holds for any $N \in \mathbb{N}^*$, if we have good estimates for C_N (we shall finally see that C_N can be controlled by C^N times a power of the factorial of N), then the theorem will follow by interpolation just as we shall do in the last section. There are two difficulties. The first one is that we have to carefully choose those operators Q^N so that the above process can go on. The second is to obtain proper estimates for C_N, which means that we have to estimate the norms of operators and remainders for every $N \in \mathbb{N}^*$ in the above process.

The paper is organized as follows. In Section 2, we introduce the spaces and give their properties we shall use. Then we construct the operator in these spaces to conjugate the original equation in Section 3. The last section is dedicated to the proof of the main theorem.

2 Definitions of operator spaces and their properties

Let us introduce some notation.

Notation 1. *We denote by* Π_n *the spectral projector on* $L^2(\mathbb{T}^d)$ *defined by*

$$\Pi_n u = \frac{e^{inx}}{(2\pi)^{d/2}}\langle u, \frac{e^{inx}}{(2\pi)^{d/2}}\rangle, \quad n \in \mathbb{Z}^d. \tag{2.1}$$

For $a \in \mathbb{R}$ *and* $b \in \mathbb{R}^d$*, we set*

$$a_+ = \max\{a, 0\}, \qquad \langle b\rangle = (1 + |b|^2)^{1/2}. \tag{2.2}$$

By $A \lesssim B$ *we mean that there is an absolute constant* $C > 0$ *such that* $A \leq CB$*. For* $s \in \mathbb{R}$*, denote by* $H^s(\mathbb{T}^d)$ *the Sobolev space consisting of* $u \in L^2(\mathbb{T}^d)$ *with its norm*

$$\|u\|_{H^s} = \Big(\sum_{n\in\mathbb{Z}^d}\langle n\rangle^{2s}\|\Pi_n u\|^2_{L^2}\Big)^{1/2} < +\infty. \tag{2.3}$$

Using the following proposition which is just Lemma 3.2 in [4], we shall give an equivalent characterization of the Sobolev space $H^s(\mathbb{T}^d)$ when $s > 0$.

Proposition 2.1 (Bourgain)**.** *Let* $\sigma \in (0, 1/10)$*. Then there are* $\tau_0 \in (0, \sigma)$*,* $\gamma > 0$ *and a partition* $(A_\alpha)_{\alpha\in\Lambda}$ *of* $\mathbb{Z}^d$ *such that*

- $\forall\alpha \in \Lambda$*,* $\forall n \in A_\alpha$*,* $\forall n' \in A_\alpha$*,* $|n - n'| + ||n|^2 - |n'|^2| < \gamma + \max(|n|, |n'|)^\sigma$*;*
- $\forall\alpha,\beta \in \Lambda$*,* $\alpha \neq \beta$*,* $\forall n \in A_\alpha$*,* $\forall n' \in A_\beta$*,* $|n - n'| + ||n|^2 - |n'|^2| > \max(|n|, |n'|)^{\tau_0}$*.*

Notation 2. *We denote for* $\alpha \in \Lambda$

$$\widetilde{\Pi}_\alpha = \sum_{n\in A_\alpha}\Pi_n.$$

For any $\alpha \in \Lambda$*, we choose* $n(\alpha) \in A_\alpha$ *and define*

$$\tilde{\Delta}u = -\sum_{\alpha\in\Lambda}|n(\alpha)|^2\widetilde{\Pi}_\alpha u. \tag{2.4}$$

By definition we know that

$$[\Delta, \tilde{\Delta}] = 0, \quad [i\partial_t, \tilde{\Delta}] = 0. \tag{2.5}$$

For $s \in \mathbb{R}$*, let* $\widetilde{H}^s(\mathbb{T}^d)$ *be the space consisting of those elements* $u \in L^2(\mathbb{T}^d)$ *with its norm*

$$\|u\|_{\widetilde{H}^s} = (\sum_{\alpha\in\Lambda}\langle n(\alpha)\rangle^{2s}\|\widetilde{\Pi}_\alpha u\|^2_{L^2})^{1/2} < +\infty. \tag{2.6}$$

By the first condition in Proposition 2.1, we deduce that there is a constant $C_0 > 0$ such that for any $s > 0$, for any $u \in \widetilde{H}^s(\mathbb{T}^d)$

$$C_0^{-s}\|u\|_{\widetilde{H}^s} \leq \|u\|_{H^s} \leq C_0^s\|u\|_{\widetilde{H}^s}. \tag{2.7}$$

We introduce some operator spaces which will be used in the next section.

Definition 2.1. *Let* $M > 0$*,* $\rho > 0$*,* $\lambda \in [1, +\infty)$*,* $\tau \in (0, 1]$*,* $\delta \in \{0, 1\}$ *and* $j \in \mathbb{N}$*. We denote by* $\mathcal{L}_\tau^{-j}(M, \rho, \lambda, \delta)$ *the space of smooth families in time of linear operators* $Q(\cdot, t)$ *from* $C^\infty(\mathbb{T}^d)$ *to* $\mathscr{D}'(\mathbb{T}^d)$ *such that there is a constant* $C > 0$ *independent of* M *and* ρ*, for which one has*

$$\begin{aligned}\|\Pi_n\partial_t^k Q(\cdot, t)\Pi_{n'}\|_{\mathcal{L}(L^2)} &\leq CM^{k+(j+\delta-1)_+}\Big[(k + (j + \delta - 1)_+)!\Big]^{\max(2,\mu)}\\ &\times e^{-\rho\,|n-n'|^{\frac{1}{\lambda}}}\langle n - n'\rangle^{-(d+2)}\Big(1 + \max(|n|, |n'|)\Big)^{-j\tau}\mathbf{1}_{\{|n-n'|\leqslant\frac{\max(|n|,|n'|)}{10(1+j)}\}}\end{aligned} \tag{2.8}$$

for any $k \in \mathbb{N}$*, any* $n, n' \in \mathbb{Z}^d$*. The best constant* C *will be denoted by* $\|Q\|^{(M,\rho,\lambda,\tau)}_{j,\delta}$*. This defines a seminorm of* $\mathcal{L}_\tau^{-j}(M, \rho, \lambda, \delta)$*.*

The notation $\|Q\|_{j,\delta}^{(M,\rho,\lambda,\tau)}$ will be abbreviated to $\|Q\|_{j,\delta}$ when M, ρ, λ, τ are fixed and there is no confusion.

Remark 2.1. *In comparison with the space introduced in Delort [4], we have added a cut-off in the definition, which depends on the size of j. This ensures that the composition of two elements in the space is essentially in the same space and the seminorm can be controlled by an absolute constant times the product of those of the original two operators. This will be described precisely in Proposition 2.7 and it is important to obtain the logarithmic growth of Sobolev norms.*

Remark 2.2. *As we shall see in Proposition 3.2, we chose the quantity* $M^{k+(j+\delta-1)_+}[(k+(j+\delta-1)_+!]^{\max(2,\mu)}$ *to ensure that all the operators which will be used to conjugate the equation (1.1) are in the same type of space, i.e.,* $\mathcal{L}_\tau^{-j}(M,\rho,\lambda,\delta)$.

Definition 2.2. *Let* $M > 0$, $\rho > 0$, $\lambda \in [1,+\infty)$, $\tau \in (0,1]$, $\delta \in \{0,1\}$ *and* $j \in \mathbb{N}$. *We denote by* $\widetilde{\mathcal{L}}_\tau^{-j}(M,\rho,\lambda,\delta)$ *the subspace of* $\mathcal{L}_\tau^{-j}(M,\rho,\lambda,\delta)$ *consisting of those elements* $Q(\cdot,t) \in \mathcal{L}_\tau^{-j}(M,\rho,\lambda,\delta)$ *such that (2.8) holds with the cut-off* $\mathbf{1}_{\{|n-n'|\leqslant \frac{\max(|n|,|n'|)}{10(1+j)}\}}$ *replaced by* $\mathbf{1}_{\{|n-n'|\leqslant \frac{\max(|n|,|n'|)}{10(2+j)}\}}$. *We also denote by* $\overline{\mathcal{L}}_\tau^{-j}(M,\rho,\lambda,\delta)$ *the set of those* $Q(\cdot,t) \in \widetilde{\mathcal{L}}_\tau^{-j}(M,\rho,\lambda,\delta)$ *such that (2.8) holds with the cut-off* $\mathbf{1}_{\{|n-n'|\leqslant \frac{\max(|n|,|n'|)}{10(1+j)}\}}$ *replaced by* $\mathbf{1}_{\{|n-n'|\leqslant \frac{\max(|n|,|n'|)}{10(2+j)},\ ||n|^2-|n'|^2|>\frac{1}{4}(|n|+|n'|)^{\tau_0}\}}$, *where* τ_0 *is given by Proposition 2.1.*

We shall also define some other convenient subspaces of $\widetilde{\mathcal{L}}_\tau^{-j}(M,\rho,\lambda,\delta)$ and $\overline{\mathcal{L}}_\tau^{-j}(M,\rho,\lambda,\delta)$.

Definition 2.3. *Let* $M > 0$, $\rho > 0$, $\lambda \in [1,+\infty)$, $\tau \in (0,1]$, $\delta \in \{0,1\}$ *and* $j \in \mathbb{N}$. *We denote by* $\widetilde{\mathcal{L}}_{\tau,D}^{-j}(M,\rho,\lambda,\delta)$ *(resp.* $\widetilde{\mathcal{L}}_{\tau,ND}^{-j}(M,\rho,\lambda,\delta)$*) the subspace of* $\widetilde{\mathcal{L}}_\tau^{-j}(M,\rho,\lambda,\delta)$ *given by those operators* $Q(\cdot,t) \in \widetilde{\mathcal{L}}_\tau^{-j}(M,\rho,\lambda,\delta)$ *such that for any* $\alpha,\beta \in \Lambda$ *with* $\alpha \neq \beta$ *(resp. any* $\alpha \in \Lambda$*)* $\widetilde{\Pi}_\alpha Q \widetilde{\Pi}_\beta \equiv 0$ *(resp.* $\widetilde{\Pi}_\alpha Q \widetilde{\Pi}_\alpha \equiv 0$*). We also set*

$$\begin{aligned}
\overline{\mathcal{L}}_{\tau,D}^{-j}(M,\rho,\lambda,\delta) &= \overline{\mathcal{L}}_\tau^{-j}(M,\rho,\lambda,\delta) \cap \widetilde{\mathcal{L}}_{\tau,D}^{-j}(M,\rho,\lambda,\delta),\\
\overline{\mathcal{L}}_{\tau,ND}^{-j}(M,\rho,\lambda,\delta) &= \overline{\mathcal{L}}_\tau^{-j}(M,\rho,\lambda,\delta) \cap \widetilde{\mathcal{L}}_{\tau,ND}^{-j}(M,\rho,\lambda,\delta).
\end{aligned}$$

Proposition 2.2. *It follows by definition that if Q is an element of* $\widetilde{\mathcal{L}}_{\tau,D}^{-j}(M,\rho,\lambda,\delta)$ *or* $\overline{\mathcal{L}}_{\tau,D}^{-j}(M,\rho,\lambda,\delta)$, *then we have* $[\tilde{\Delta}, Q] = 0$.

Notation 3. *Let* $M > 0$, $\rho > 0$, $\lambda \in [1,+\infty)$, $\tau \in (0,1]$, $\delta \in \{0,1\}$ *and* $j \in \mathbb{N}$. *If Q is an element of* $\mathcal{L}_\tau^{-j}(M,\rho,\lambda,\delta)$ *(resp.* $\widetilde{\mathcal{L}}_\tau^{-j}(M,\rho,\lambda,\delta)$, $\overline{\mathcal{L}}_\tau^{-j}(M,\rho,\lambda,\delta)$*), we denote*

$$Q_D = \sum_{\alpha\in\Lambda} \widetilde{\Pi}_\alpha Q \widetilde{\Pi}_\alpha, \quad Q_{ND} = \sum_{\substack{\alpha,\beta\in\Lambda\\ \alpha\neq\beta}} \widetilde{\Pi}_\alpha Q \widetilde{\Pi}_\beta. \tag{2.9}$$

By definition we immediately have

$$\begin{gathered}
\|Q_D\|_{j,\delta} \le \|Q\|_{j,\delta}, \quad \|Q_{ND}\|_{j,\delta} \le \|Q\|_{j,\delta},\\
Q_D \in \mathcal{L}_{\tau,D}^{-j}(M,\rho,\lambda,\delta) \text{ (resp. } \widetilde{\mathcal{L}}_{\tau,D}^{-j}(M,\rho,\lambda,\delta),\ \overline{\mathcal{L}}_{\tau,D}^{-j}(M,\rho,\lambda,\delta)),\\
Q_{ND} \in \mathcal{L}_{\tau,ND}^{-j}(M,\rho,\lambda,\delta) \text{ (resp. } \widetilde{\mathcal{L}}_{\tau,ND}^{-j}(M,\rho,\lambda,\delta),\ \overline{\mathcal{L}}_{\tau,ND}^{-j}(M,\rho,\lambda,\delta)).
\end{gathered} \tag{2.10}$$

Proposition 2.3. *Let* $M > 0$, $\rho > 0$, $\lambda \in [1,+\infty)$, $\tau \in (0,\tau_0]$, $\delta \in \{0,1\}$ *and* $j \in \mathbb{N}^*$. *Here* τ_0 *is given by Proposition 2.1. Assume* $S \in \overline{\mathcal{L}}_{\tau,ND}^{-(j-1)}(M,\rho,\lambda,\delta)$. *Then the equation* $[Q,\Delta] = -S$ *defines an element* $Q \in \mathcal{L}_\tau^{-j}(M,\rho,\lambda,0)$ *with* $\|Q\|_{j,0} \lesssim \|S\|_{j-1,\delta}$. *If S is self-adjoint, then* $Q^* = -Q$, *where* Q^* *denote the adjoint of Q (at fixed time, for the usual* L^2*-pairing).*

Proof. The equation $[Q, \Delta] = -S$ may be written

$$(|n'|^2 - |n|^2)\Pi_n Q \Pi_{n'} = \Pi_n S \Pi_{n'}. \tag{2.11}$$

To define $Q \in \mathcal{L}_\tau^{-j}(M, \rho, \lambda, 0)$, we only need to estimate $\|\Pi_n Q \Pi_{n'}\|_{\mathcal{L}(L^2)}$ when it is non zero. So we may assume both sides of (2.11) are non zero. Since $S \in \overline{\mathcal{L}}_{\tau,ND}^{-(j-1)}(M, \rho, \lambda, \delta)$, we then have

$$|\, n - n'| \leqslant \frac{\max(|n|, |n'|)}{10(1 + j)}, \quad \big||n|^2 - |n'|^2\big| > \frac{1}{4}(|n| + |n'|)^{\tau_0},$$

which, together with (2.11) and the fact $\tau \leq \tau_0$, allows us to deduce

$$\begin{aligned}\|\Pi_n Q \Pi_{n'}\|_{\mathcal{L}(L^2)} \lesssim \|S\|_{j-1,\delta} M^{k+(j-1)_+} \Big[(k + (j-1)_+)!\Big]^{\max(2,\mu)} \\ \times e^{-\rho\, |\, n-n'|^{\frac{1}{\lambda}}} \langle n - n' \rangle^{-(d+2)} \Big(1 + \max(|n|, |n'|)\Big)^{-j\tau} \mathbf{1}_{\{|\, n-n'| \leqslant \frac{\max(|n|,|n'|)}{10(1+j)}\}}.\end{aligned}$$

This means $Q \in \mathcal{L}_\tau^{-j}(M, \rho, \lambda, 0)$ and $\|Q\|_{j,0} \lesssim \|S\|_{j-1,\delta}$. If S is self-adjoint, then by (2.11) we see that $Q^* = -Q$. This concludes the proof. □

We shall also need the following remainder operators which raise the order of regularity as much as we want.

Definition 2.4. *Let $M > 0$, $\tau \in (0, 1]$ and $j \in \mathbb{N}$. We denote by $\mathcal{R}_j^{-\infty}(M, \tau)$ the space of smooth families in time of linear operators $R(\cdot, t)$ from $C^\infty(\mathbb{T}^d)$ to $\mathscr{D}'(\mathbb{T}^d)$ such that there is a constant $C > 0$ independent of M, for which one has*

$$\begin{aligned}\|\Pi_n \partial_t^k R(\cdot, t)\Pi_{n'}\|_{\mathcal{L}(L^2)} \leq C M^{N+j+k}((j+k)!)^{\max(2,\mu)} N! \\ \times \langle n - n' \rangle^{-(d+2)} \Big(1 + \max(|n|, |n'|)\Big)^{-\tau N}\end{aligned} \tag{2.12}$$

for any $k \in \{0, 1\}$, any $N \in \mathbb{N}$, any $n, n' \in \mathbb{Z}^d$. The best constant C will be denoted by $|R|_j^{(M,\tau)}$. This defines a seminorm of $\mathcal{R}_j^{-\infty}(M, \tau)$.

Similarly as before, the notation $|R|_j^{(M,\tau)}$ will be abbreviated to $|R|_j$ when M, τ are fixed and there is no confusion.

By definition, we immediately have the following proposition.

Proposition 2.4. *Let $M > 1$, $\rho > 0$, $\lambda \in [1, +\infty)$, $\tau \in (0, 1]$ and $j \in \mathbb{N}^*$. If $Q \in \mathcal{L}_\tau^{-j}(M, \rho, \lambda, 0)$, then*

$$[i\partial_t, Q] = i\partial_t Q \in \mathcal{L}_\tau^{-j}(M, \rho, \lambda, 1) \quad \text{and} \quad \|[i\partial_t, Q]\|_{j,1} \leq \|Q\|_{j,0}. \tag{2.13}$$

The elements defined in the above definitions may be extended as bounded linear operators acting on Sobolev spaces.

Proposition 2.5. *Let $M > 0$, $\rho > 0$, $\lambda \in [1, +\infty)$, $\tau \in (0, 1]$, $\delta \in \{0, 1\}$ and $j \in \mathbb{N}$. Let $Q \in \mathcal{L}_\tau^{-j}(M, \rho, \lambda, \delta)$. Then for any $k \in \mathbb{N}$, $\partial_t^k Q$ extends as a bounded linear operator from $H^s(\mathbb{T}^d)$ to $H^{s+j\tau}(\mathbb{T}^d)$ for any $s \in \mathbb{R}$. Moreover, its operator norm, denoted by $\|\partial_t^k Q\|_{\mathcal{L}(H^s, H^{s+j\tau})}$, satisfies*

$$\|\partial_t^k Q\|_{\mathcal{L}(H^s, H^{s+j\tau})} \lesssim C_1^{|s|} \|Q\|_{j,\delta} M^{k+(j+\delta-1)_+} \Big((k + (j+\delta-1)_+)!\Big)^{\max(2,\mu)}, \tag{2.14}$$

where $C_1 > 1$ is an absolute constant. Recall that by $A \lesssim B$ we mean that there is a constant C independent of any other quantities such that $A \leq CB$.

Proof. Assume $u \in H^s(\mathbb{T}^d)$. Since $|n-n'| \leqslant \frac{\max(|n|,|n'|)}{10(1+j)}$ implies $C_1^{-1}\langle n'\rangle \le \langle n\rangle \le C_1\langle n'\rangle$ for some absolute constant C_1, we compute using (2.8)

$$\begin{aligned}
\|\partial_t^k Qu\|^2_{H^{s+j\tau}} &= \sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2(s+j\tau)} \|\Pi_n \partial_t^k Qu\|^2_{L^2} \\
&\le \sum_{n\in\mathbb{Z}^d}\Big(\sum_{n'\in\mathbb{Z}^d}\langle n\rangle^{s+j\tau}\|\Pi_n\partial_t^k Q\Pi_{n'}u\|_{L^2}\Big)^2 \\
&\le \sum_{n\in\mathbb{Z}^d}\Big(\sum_{n'\in\mathbb{Z}^d}\langle n\rangle^{s+j\tau}\|Q\|_{j,\delta}M^{k+(j+\delta-1)_+}[(k+(j+\delta-1)_+)!]^{\max(2,\mu)} \\
&\qquad \times \langle n-n'\rangle^{-(d+2)}(1+\max(|n|,|n'|))^{-j\tau}\mathbf{1}_{\{|n-n'|\leqslant\frac{\max(|n|,|n'|)}{10(1+j)}\}}\|\Pi_{n'}u\|_{L^2}\Big)^2 \\
&\le C_1^{2|s|}\|Q\|^2_{j,\delta}M^{2(k+(j+\delta-1)_+)}[(k+(j+\delta-1)_+)!]^{2\max(2,\mu)} \\
&\qquad \times \sum_{n\in\mathbb{Z}^d}\Big(\sum_{n'\in\mathbb{Z}^d}\langle n-n'\rangle^{-(d+2)}\langle n'\rangle^{s}\|\Pi_{n'}u\|_{L^2}\Big)^2 \\
&\lesssim C_1^{2|s|}\|Q\|^2_{j,\delta}M^{2(k+(j+\delta-1)_+)}[(k+(j+\delta-1)_+)!]^{2\max(2,\mu)}\|u\|^2_{H^s},
\end{aligned}$$

where in the last step we used Young inequality. The conclusion follows by taking the square root of both sides. □

Proposition 2.6. *Let $M>0$, $\tau\in(0,1]$ and $j\in\mathbb{N}$. Let $R\in\mathcal{R}_j^{-\infty}(M,\tau)$. Then operators $R(\cdot,t)$, $\partial_t R$ and $[\Delta,R]$ may be extended as bounded linear operators from $H^{-s}(\mathbb{T}^d)$ to $H^{-s'+\tau m}(\mathbb{T}^d)$ for any $s,s'\ge 0$ and any $m\in\mathbb{N}$. Moreover, for any $k\in\{0,1\}$*

$$\begin{aligned}
\|\partial_t^k R\|_{\mathcal{L}(H^{-s},H^{-s'+\tau m})} &\lesssim |R|_j M^{m+[\frac{s+1}{\tau}]+j+k}((j+k)!)^{\max(2,\mu)}(m+[\frac{s+1}{\tau}])!, \\
\|[\Delta,R]\|_{\mathcal{L}(H^{-s},H^{-s'+\tau m})} &\lesssim |R|_j M^{m+[\frac{s+2}{\tau}]+j}(j!)^{\max(2,\mu)}(m+[\frac{s+2}{\tau}])!,
\end{aligned} \tag{2.15}$$

where $[\cdot]$ means the integer part of a real number.

Proof. Let $s\ge 0$, $s'\ge 0$, $m\in\mathbb{N}$, $u\in H^{-s}(\mathbb{T}^d)$. For $k\in\{0,1\}$, we have by (2.12) with $N=m+[\frac{s+1}{\tau}]$

$$\begin{aligned}
\|\partial_t^k Ru\|^2_{H^{-s'+\tau m}} &= \sum_{n\in\mathbb{Z}^d}\langle n\rangle^{-2s'+2\tau m}\|\Pi_n\partial_t^k Ru\|^2_{L^2} \\
&\le \sum_{n\in\mathbb{Z}^d}\Big[\sum_{n'\in\mathbb{Z}^d}\langle n\rangle^{-s'+\tau m}|R|_j M^{m+[\frac{s+1}{\tau}]+j+k}((j+k)!)^{\max(2,\mu)}(m+[\frac{s+1}{\tau}])! \\
&\qquad \times\langle n-n'\rangle^{-(d+2)}(1+\max(|n|,|n'|))^{-\tau(m+[\frac{s+1}{\tau}])}\|\Pi_{n'}u\|_{L^2}\Big]^2 \\
&\le |R|_j^2 M^{2(m+[\frac{s+1}{\tau}]+j+k)}((j+k)!)^{2\max(2,\mu)}[(m+[\frac{s+1}{\tau}])!]^2 \\
&\qquad \times\sum_{n\in\mathbb{Z}^d}\Big[\sum_{n'\in\mathbb{Z}^d}\langle n-n'\rangle^{-(d+2)}\langle n'\rangle^{-s}\|\Pi_{n'}u\|_{L^2}\Big]^2 \\
&\lesssim |R|_j^2 M^{2(m+[\frac{s+1}{\tau}]+j+k)}((j+k)!)^{2\max(2,\mu)}[(m+[\frac{s+1}{\tau}])!]^2\|u\|^2_{H^{-s}},
\end{aligned}$$

where in the last step we used Young inequality. The first inequality of (2.15) follows by taking the square root of both sides. The second inequality follows by a similar argument and by noting that $||n|^2-|n'|^2|\lesssim\langle n-n'\rangle(1+\max(|n|,|n'|))$ and taking $N=m+[\frac{s+2}{\tau}]$ in (2.12). □

When one conjugates the original equation, one needs to compute the composition of two elements in $\mathcal{L}_\tau^{-j}(M,\rho,\lambda,\delta)$ and the commutator $[i\partial_t,Q]$ for $Q\in\mathcal{L}_\tau^{-j}(M,\rho,\lambda,0)$. First of all let us introduce some notation before we give a precise description of that.

Notation 4. *Recall that Q^* denote the adjoint of $Q \in \mathcal{L}_\tau^{-j}(M,\rho,\lambda,\delta)$ ($\delta \in \{0,1\}$, at fixed time, for the usual L^2-pairing). If $Q_i \in \mathcal{L}_\tau^{-j_i}(M,\rho,\lambda,\delta_i)$, $j_i \in \mathbb{N}$, $\delta_i \in \{0,1\}$, $i = 1,2$, we then denote*

$$\begin{aligned}\mathscr{M}(Q_1,Q_2) &= \sum_{n,\,n'\in\mathbb{Z}^d} \Pi_n(Q_1\circ Q_2)\Pi_{n'}\boldsymbol{1}_{\{|n-n'|\leqslant \frac{\max(|n|,|n'|)}{10(2+j_1+j_2)}\}}\,,\\ \mathscr{R}(Q_1,Q_2) &= \sum_{n,\,n'\in\mathbb{Z}^d} \Pi_n(Q_1\circ Q_2)\Pi_{n'}\boldsymbol{1}_{\{|n-n'|> \frac{\max(|n|,|n'|)}{10(2+j_1+j_2)}\}}\end{aligned} \tag{2.16}$$

We shall also denote

$$\begin{aligned}\mathscr{M}'(Q_1,Q_2) &= \mathscr{M}(Q_1,Q_2) + [\mathscr{M}(Q_1,Q_2)]^*,\\ \mathscr{R}'(Q_1,Q_2) &= \mathscr{R}(Q_1,Q_2) + [\mathscr{R}(Q_1,Q_2)]^*.\end{aligned} \tag{2.17}$$

The operator $\mathscr{M}(Q_1,Q_2)$ is the main part of the operator obtained by composing Q_1 and Q_2. As we shall see, it essentially falls into the same operator class as the original ones. The remainder part, i.e., $\mathscr{R}(Q_1,Q_2)$ is a regularizing operator. Moreover, $\mathscr{M}'(Q_1,Q_2)$ and $\mathscr{R}'(Q_1,Q_2)$ are obviously self-adjoint. Remark that for $\rho > 0$, $\lambda \in [1,+\infty)$, $\tau \in (0,1]$, denoting

$$\theta_0(\rho,\lambda,\tau) = \min\Big(\frac{2}{60^{1/(2\lambda)}}\big(\log\frac{162}{100}\big)^{\frac12}(\rho\tau)^{\frac12}, \frac{\rho}{(30\sqrt{2}\,)^{1/\lambda}}\Big), \tag{2.18}$$

we have that

$$\exp\{-\rho(\frac{x}{10(2+t)})^{\frac1\lambda}\}\,(1+x)^{-t\tau}(\frac{100}{81})^{t\tau} \le \exp\{-\theta_0(\rho,\lambda,\tau)(x+1)^{\frac{1}{2\lambda}}\} \tag{2.19}$$

holds for any $x \ge 1$, any $t \ge 0$. Denote

$$\theta_1(\rho,\tau) = 1 + \max_{\lambda\ge1}\,[\theta_0(\rho,\lambda,\tau)]^{-1}. \tag{2.20}$$

Proposition 2.7. *Let $\rho > 0$, $\lambda \in [1,+\infty)$, $\tau \in (0,1]$ and $j_1, j_2 \in \mathbb{N}$. Let $M > \theta_1(\rho,\tau)$ and $j = j_1 + j_2$. Assume $Q_1 \in \mathcal{L}_\tau^{-j_1}(M,\rho,\lambda,0)$ and $Q_2 \in \mathcal{L}_\tau^{-j_2}(M,\rho,\lambda,0)$. Then one has*

$$Q_1\circ Q_2 = \mathscr{M}(Q_1,Q_2) + \mathscr{R}(Q_1,Q_2) \tag{2.21}$$

with

$$\begin{aligned}\mathscr{M}(Q_1,Q_2) &\in \widetilde{\mathcal{L}}_\tau^{-j}(M,\rho,\lambda,0), &\quad \|\mathscr{M}(Q_1,Q_2)\|_{j,0} &\lesssim \|Q_1\|_{j_1,0}\|Q_2\|_{j_2,0},\\ \mathscr{R}(Q_1,Q_2) &\in \mathcal{R}_j^{-\infty}(M,\frac{1}{2\lambda}), &\quad |\mathscr{R}(Q_1,Q_2)|_j &\lesssim \|Q_1\|_{j_1,0}\|Q_2\|_{j_2,0}.\end{aligned} \tag{2.22}$$

Proof. We only need to check (2.22). For $k \in \mathbb{N}$, we have by (2.8)

$$\begin{aligned}&\|\Pi_n\partial_t^k\mathscr{M}(Q_1,Q_2)\Pi_{n'}\|_{\mathcal{L}(L^2)}\\ \le& \sum_{k_1+k_2=k}\binom{k}{k_1}\sum_{\ell\in\mathbb{Z}^d}\|\Pi_n\partial_t^{k_1}Q_1\Pi_\ell\|_{\mathcal{L}(L^2)}\|\Pi_\ell\partial_t^{k_2}Q_2\Pi_{n'}\|_{\mathcal{L}(L^2)}\mathbf{1}_{\{|n-n'|\leqslant\frac{\max(|n|,|n'|)}{10(2+j)}\}}\\ \le& \sum_{k_1+k_2=k}\sum_{\ell\in\mathbb{Z}^d}\binom{k}{k_1}\|Q_1\|_{j_1,0}\|Q_2\|_{j_2,0}M^{k+(j_1-1)_++(j_2-1)_+}\mathbf{1}_{\{|n-n'|\leqslant\frac{\max(|n|,|n'|)}{10(2+j)}\}}\\ &\times\Big[(k_1+(j_1-1)_+)!\Big]^{\max(2,\mu)}\Big[(k_2+(j_2-1)_+)!\Big]^{\max(2,\mu)}\\ &\times e^{-\rho|n-n'|^{\frac1\lambda}}\langle n-\ell\rangle^{-(d+2)}\langle\ell-n'\rangle^{-(d+2)}\\ &\times\Big(1+\max(|n|,|\ell|)\Big)^{-j_1\tau}\Big(1+\max(|\ell|,|n'|)\Big)^{-j_2\tau}\\ &\times\mathbf{1}_{\{|n-\ell|\leqslant\frac{\max(|n|,|\ell|)}{10(1+j_1)}\}}\mathbf{1}_{\{|n'-\ell|\leqslant\frac{\max(|n'|,|\ell|)}{10(1+j_2)}\}}\mathbf{1}_{\{|n-n'|\leqslant\frac{\max(|n|,|n'|)}{10(2+j)}\}},\end{aligned} \tag{2.23}$$

where we have use the following inequality:

$$|n-\ell|^{\frac{1}{\lambda}} + |\ell - n'|^{\frac{1}{\lambda}} \geq |n-n'|^{\frac{1}{\lambda}} \quad \text{when } \lambda \geq 1.$$

We need to estimate the following two terms:

$$\mathbf{I} \overset{def}{=} \sum_{k_1+k_2=k} \binom{k}{k_1} \Big[(k_1 + (j_1-1)_+)!\Big]^{\max(2,\mu)} \Big[(k_2 + (j_2-1)_+)!\Big]^{\max(2,\mu)},$$

$$\mathbf{II} \overset{def}{=} \text{the last two lines of (2.23).}$$

To obtain an estimate for $\mathbf{I}$, let us first estimate

$$\mathbf{I}' = \sum_{k_1+k_2=k} \binom{k}{k_1} \Big[(k_1 + (j_1-1)_+)!\Big]^2 \Big[(k_2 + (j_2-1)_+)!\Big]^2.$$

If neither of $(j_1-1)_+$ and $(j_2-1)_+$ is larger than 0, then

$$\mathbf{I}' = \sum_{k_1+k_2=k} k!k_1!k_2! \leq 3(k!)^2.$$

If only one of $(j_1-1)_+$ and $(j_2-1)_+$ is larger than 0, for instance, $(j_2-1)_+ > 0$, then

$$\begin{aligned}
\mathbf{I}' &= \sum_{k_1+k_2=k} \binom{k}{k_1} [k_1!]^2 [(k_2+j_2-1)!]^2 \\
&\leq 2[(k+j-1)!]^2 + \sum_{\substack{k_1+k_2=k \\ k_1\geq 1, k_2 \geq 1}} k!k_1!(k_2+j_2-1+k_1-1)\ldots(2+k_1-1) \\
&\qquad\qquad \times (k_2+j_2-1+k_1)\ldots(k_2+1+k_1) \\
&\leq 3[(k+j-1)!]^2,
\end{aligned}$$

while if both of $(j_1-1)_+$ and $(j_2-1)_+$ are larger than 0, then

$$\begin{aligned}
\mathbf{I}' &= \sum_{k_1+k_2=k} \binom{k}{k_1} [(k_1+j_1-1)!]^2 [(k_2+j_2-1)!]^2 \\
&= \sum_{k_1+k_2=k} k!(k_1+j_1-1)!(k_2+j_2-1)! \\
&\qquad \times (k_1+j_1-1)\ldots(k_1+1)(k_2+j_2-1)\ldots(k_2+1) \\
&\leq \sum_{k_1+k_2=k} k!(k_1+j_1-1+k_2+j_2-1)\ldots(1+k_2+j_2-1)(k_2+j_2-1)! \\
&\qquad \times (k_1+j_1-1+k_2+j_2-1)\ldots(k_1+1+k_2+j_2-1) \\
&\qquad \times (k_2+j_2-1+k_1)\ldots(k_2+1+k_1) \\
&\leq [(k+j-1)!]^2.
\end{aligned}$$

Thus we always have

$$\mathbf{I}' \leq 3[(k+(j-1)_+)!]^2. \tag{2.24}$$

Since

$$\Big[(k_1+(j_1-1)_+)!\Big]^{(\mu-2)_+} \Big[(k_2+(j_2-1)_+)!\Big]^{(\mu-2)_+} \leq [(k+(j-1)_+)!]^{(\mu-2)_+},$$

we have by (2.24)

$$\mathbf{I} \leq 3[(k+(j-1)_+)!]^{\max(2,\mu)}. \tag{2.25}$$

We first assume $|n'| \geq |n|$ when estimating $\mathbf{II}$. From the cut-offs we deduce $|n'| \leq 2|n|$ so that

$$|n - n'| \leq \frac{\max(|n|, |n'|)}{10(2 + j)} = \frac{|n'|}{10(2 + j)} \leq \frac{|n|}{5(1 + j)}.$$

Therefore

$$\begin{aligned}
\mathbf{II} &\leq (1 + |n|)^{-j_1\tau}(1 + |n'|)^{-j_2\tau}\mathbf{1}_{\{|n-n'|\leqslant \frac{|n|}{5(1+j)}\}} \\
&\leq (1 + |n'|)^{-j\tau}(1 + \frac{|n - n'|}{1 + |n|})^{j_1\tau}\mathbf{1}_{\{|n-n'|\leqslant \frac{|n|}{5(1+j)}\}} \\
&\leq (1 + |n'|)^{-j\tau}(1 + \frac{1}{5(1 + j)})^{j\tau} \\
&\leq 3(1 + |n'|)^{-j\tau}.
\end{aligned}$$

We may get an analogue when $|n| \geq |n'|$ and thus we obtain

$$\mathbf{II} \leq 3(1 + \max(|n|, |n'|))^{-j\tau}. \tag{2.26}$$

Plugging (2.25), (2.26) into (2.23) and using the fact that

$$\sum_{\ell\in\mathbb{Z}^d} \langle n - \ell\rangle^{-(d+2)}\langle \ell - n'\rangle^{-(d+2)} \lesssim \langle n - n'\rangle^{-(d+2)}, \tag{2.27}$$

we obtain

$$\begin{aligned}
\|\Pi_n\partial_t^k\mathscr{M}(Q_1, Q_2)\Pi_{n'}\|_{\mathcal{L}(L^2)} &\lesssim \|Q_1\|_{j_1,0}\|Q_2\|_{j_2,0}M^{k+(j-1)_+}[(k + (j - 1)_+)!]^{\max(2,\mu)} \\
&\quad \times e^{-\rho|n-n'|^{\frac{1}{\lambda}}}\langle n - n'\rangle^{-(d+2)}(1 + \max(|n|, |n'|))^{-j\tau}\mathbf{1}_{\{|n-n'|\leqslant \frac{\max(|n|,|n'|)}{10(2+j)}\}},
\end{aligned}$$

which implies the claims in the first line of (2.22).

We are left with estimating the remainder operator. We have for $k \in \{0, 1\}$

$$\begin{aligned}
&\|\Pi_n\partial_t^k\mathscr{R}(Q_1, Q_2)\Pi_{n'}\|_{\mathcal{L}(L^2)} \\
\leq &\sum_{k_1+k_2=k}\sum_{\ell\in\mathbb{Z}^d}\binom{k}{k_1}\|\Pi_n\partial_t^{k_1}Q_1\Pi_\ell\|_{\mathcal{L}(L^2)}\|\Pi_\ell\partial_t^{k_2}Q_2\Pi_{n'}\|_{\mathcal{L}(L^2)}\mathbf{1}_{\{|n-n'|> \frac{\max(|n|,|n'|)}{10(2+j)}\}} \\
\leq &\sum_{k_1+k_2=k}\sum_{\ell\in\mathbb{Z}^d}\binom{k}{k_1}\|Q_1\|_{j_1,0}\|Q_2\|_{j_2,0}M^{k+(j_1-1)_++(j_2-1)_+}\mathbf{1}_{\{|n-n'|> \frac{\max(|n|,|n'|)}{10(2+j)}\}} \\
&\times\Big[(k_1 + (j_1 - 1)_+)!\Big]^{\max(2,\mu)}\Big[(k_2 + (j_2 - 1)_+)!\Big]^{\max(2,\mu)} \\
&\times e^{-\rho|n-n'|^{\frac{1}{\lambda}}}\langle n - \ell\rangle^{-(d+2)}\langle \ell - n'\rangle^{-(d+2)} \\
&\times\Big(1 + \max(|n|, |\ell|)\Big)^{-j_1\tau}\Big(1 + \max(|\ell|, |n'|)\Big)^{-j_2\tau} \\
&\times\mathbf{1}_{\{|n-\ell|\leqslant \frac{\max(|n|,|\ell|)}{10(1+j_1)}\}}\mathbf{1}_{\{|n'-\ell|\leqslant \frac{\max(|n'|,|\ell|)}{10(1+j_2)}\}}\mathbf{1}_{\{|n-n'|> \frac{\max(|n|,|n'|)}{10(2+j)}\}}.
\end{aligned} \tag{2.28}$$

We only deal with the case $|n'| \geq |n|$. The other will be the same. Thus we assume

$$\max(|n|, |n'|) = |n'|. \tag{2.29}$$

We denote by $\mathbf{III}$ the last two lines of (2.28). We may assume that $\mathbf{III}$ is non-zero when estimating it. Thus by the cut-offs we deduce

$$|n| \geq \frac{9}{10}|\ell| \geq \frac{81}{100}|n'| > 0$$

so that

$$\mathbf{III} \leq (1+|n|)^{-j_1\tau}(1+|n'|)^{-j_2\tau}\mathbf{1}_{\{\frac{81}{100}|n'|\leq|n|\}} \leq (1+|n'|)^{-j\tau}(\frac{100}{81})^{j_1\tau}.$$

Thus by the assumption (2.29), (2.19), (2.20) and the assumption on M, we have

$$\begin{aligned} e^{-\rho|n-n'|^{\frac{1}{\lambda}}}\mathbf{1}_{\{|n-n'|>\frac{\max(|n|,|n'|)}{10(2+j)}\}}\mathbf{III} &\leq e^{-\rho|n-n'|^{\frac{1}{\lambda}}}(1+|n'|)^{-j\tau}(\frac{100}{81})^{j_1\tau}\mathbf{1}_{\{|n-n'|>\frac{|n'|}{10(2+j)}\}} \\ &\leq e^{-\rho(\frac{|n'|}{10(2+j)})^{\frac{1}{\lambda}}}(1+|n'|)^{-j\tau}(\frac{100}{81})^{j\tau} \\ &\leq e^{-\theta_0(\rho,\lambda,\tau)(|n'|+1)^{\frac{1}{2\lambda}}} \\ &\leq (\frac{1}{\theta_0(\rho,\lambda,\tau)})^N N!\,(1+\max(|n|,|n'|))^{-\frac{N}{2\lambda}} \\ &\leq M^N N!\,(1+\max(|n|,|n'|))^{-\frac{N}{2\lambda}}. \end{aligned} \tag{2.30}$$

Plugging (2.30), (2.27), (2.25) into (2.28), we obtain for $k \in \{0,1\}$ and for any $N \in \mathbb{N}$

$$\begin{aligned} \|\Pi_n\partial_t^k\mathscr{R}(Q_1,Q_2)\Pi_{n'}\|_{\mathcal{L}(L^2)} \lesssim \|Q_1\|_{j_1,0}\|Q_2\|_{j_2,0}M^{N+k+j}((j+k)!)^{\max(2,\mu)}N! \\ \times\langle n-n'\rangle^{-(d+2)}(1+\max(|n|,|n'|))^{-\frac{N}{2\lambda}}, \end{aligned}$$

which gives the claims in the second line of (2.22) and concludes the proof. □

We also have the following proposition.

Proposition 2.8. *Let $\rho>0$, $\lambda\in[1,+\infty)$, $\tau\in(0,1]$ and $j_1, j_2\in\mathbb{N}^*$. Let $M>\theta_1(\rho,\tau)$ and $j=j_1+j_2$. Assume $Q_1\in\mathcal{L}_\tau^{-j_1}(M,\rho,\lambda,0)$ and $Q_2\in\mathcal{L}_\tau^{-j_2}(M,\rho,\lambda,1)$. Then one has*

$$\begin{aligned} Q_1\circ Q_2 &= \mathscr{M}(Q_1,Q_2)+\mathscr{R}(Q_1,Q_2), \\ Q_2\circ Q_1 &= \mathscr{M}(Q_2,Q_1)+\mathscr{R}(Q_2,Q_1), \end{aligned} \tag{2.31}$$

with

$$\begin{aligned} \mathscr{M}(Q_1,Q_2),\mathscr{M}(Q_2,Q_1) &\in \widetilde{\mathcal{L}}_\tau^{-j}(M,\rho,\lambda,0), \\ \mathscr{R}(Q_1,Q_2),\mathscr{R}(Q_2,Q_1) &\in \mathcal{R}_j^{-\infty}(M,\frac{1}{2\lambda}), \\ \|\mathscr{M}(Q_1,Q_2)\|_{j,0}+\|\mathscr{M}(Q_1,Q_2)\|_{j,0} &\lesssim \|Q_1\|_{j_1,0}\|Q_2\|_{j_2,1}, \\ |\mathscr{R}(Q_1,Q_2)|_j+|\mathscr{R}(Q_2,Q_1)|_j &\lesssim \|Q_1\|_{j_1,0}\|Q_2\|_{j_2,1}. \end{aligned} \tag{2.32}$$

Proof. The proof is the same as that of Proposition 2.7 except that instead of estimating $\mathbf{I}$, we have to estimate

$$\mathbf{I}'' \stackrel{def}{=} \sum_{k_1+k_2=k}\binom{k}{k_1}\Big((k_1+j_1-1)!\Big)^{\max(2,\mu)}\Big((k_2+j_2)!\Big)^{\max(2,\mu)},$$

which is less or equals $3[(k+(j-1)_+)!]^{\max(2,\mu)}$ if $j_1, j_2\in\mathbb{N}^*$. Note that $M^{k+(j_1-1)_++j_2}\leq M^{k+(j-1)_+}$ fails when $j_1=0$ and $j_2\in\mathbb{N}^*$. However, we shall only need to use the result for $j_1, j_2\in\mathbb{N}^*$. □

The following two corollaries are immediate consequences of Proposition 2.7 and 2.8.

Corollary 2.1. *Under the hypotheses of Proposition 2.7, one has*

$$Q_1 \circ Q_2 + (Q_1 \circ Q_2)^* = \mathscr{M}'(Q_1, Q_2) + \mathscr{R}'(Q_1, Q_2). \tag{2.33}$$

Moreover, $\mathscr{M}'(Q_1, Q_2)$, $\mathscr{R}'(Q_1, Q_2)$ are self-adjoint and we have

$$\begin{aligned} \mathscr{M}'(Q_1, Q_2) &\in \widetilde{\mathcal{L}}_\tau^{-j}(M, \rho, \lambda, 0), \quad \mathscr{R}'(Q_1, Q_2) \in \mathcal{R}_j^{-\infty}(M, \frac{1}{2\lambda}), \\ \|\mathscr{M}'(Q_1, Q_2)\|_{j,0} &\lesssim \Big(\|Q_1\|_{j_1,0}\|Q_2\|_{j_2,0} + \|Q_1^*\|_{j_1,0}\|Q_2^*\|_{j_2,0}\Big), \\ |\mathscr{R}'(Q_1, Q_2)|_j &\lesssim \Big(\|Q_1\|_{j_1,0}\|Q_2\|_{j_2,0} + \|Q_1^*\|_{j_1,0}\|Q_2^*\|_{j_2,0}\Big). \end{aligned} \tag{2.34}$$

Corollary 2.2. *Under the hypotheses of Proposition 2.8, one has*

$$\begin{aligned} Q_1 \circ Q_2 + (Q_1 \circ Q_2)^* &= \mathscr{M}'(Q_1, Q_2) + \mathscr{R}'(Q_1, Q_2), \\ Q_2 \circ Q_1 + (Q_2 \circ Q_1)^* &= \mathscr{M}'(Q_2, Q_1) + \mathscr{R}'(Q_2, Q_1). \end{aligned} \tag{2.35}$$

Moreover, $\mathscr{M}'(Q_1, Q_2)$, $\mathscr{R}'(Q_1, Q_2)$ are self-adjoint and (2.34) *holds. $\mathscr{M}'(Q_2, Q_1)$, $\mathscr{R}'(Q_2, Q_1)$ respectively have the same properties as that of $\mathscr{M}'(Q_1, Q_2)$, $\mathscr{R}'(Q_1, Q_2)$.*

Proposition 2.9. *Let $\rho > 0$, $\lambda \in [1, +\infty)$, $\tau \in (0, 1]$, $M > \theta_1(\rho, \tau)$ and $j \in \mathbb{N}^*$. Let $Q \in \mathcal{L}_\tau^{-j}(M, \rho, \lambda, 1)$. Then one may decompose*

$$Q = \widetilde{Q} + \widetilde{R} \tag{2.36}$$

with

$$\begin{aligned} \widetilde{Q} &\in \widetilde{\mathcal{L}}_\tau^{-j}(M, \rho, \lambda, 1), \quad \|\widetilde{Q}\|_{j,1} \le \|Q\|_{j,1}, \\ \widetilde{R} &\in \mathcal{R}_j^{-\infty}(M, \frac{1}{2\lambda}), \quad |\widetilde{R}|_j \le \|Q\|_{j,1}. \end{aligned} \tag{2.37}$$

Moreover, if we further assume that Q is a self-adjoint operator (for fixed t, Q extends as a bounded linear operator on $L^2(\mathbb{T}^d)$ by Proposition 2.5), so are $\widetilde{Q}$ and $\widetilde{R}$.

Proof. Defining

$$\begin{aligned} \widetilde{Q} &= \sum_n \sum_{n'} \Pi_n Q \Pi_{n'} \mathbf{1}_{\{|n-n'| \leqslant \frac{\max(|n|,|n'|)}{10(2+j)}\}}, \\ \widetilde{R} &= \sum_n \sum_{n'} \Pi_n Q \Pi_{n'} \mathbf{1}_{\{|n-n'| > \frac{\max(|n|,|n'|)}{10(2+j)}\}}, \end{aligned}$$

we see that (2.36) holds and that the claims in the first line of (2.37) hold true. For $k \in \{0, 1\}$, any $N \in \mathbb{N}$, we have by (2.19) and (2.20)

$$\begin{aligned} &\|\Pi_n \partial_t^k \widetilde{R} \Pi_{n'}\|_{\mathcal{L}(L^2)} \\ &\le \|Q\|_{j,1} M^{k+j} ((k+j)!)^{\max(2,\mu)} \langle n - n' \rangle^{-(d+2)} \\ &\qquad \times e^{-\rho |n-n'|^{\frac{1}{\lambda}}} (1 + \max(|n|, |n'|))^{-j\tau} \mathbf{1}_{\{\frac{\max(|n|,|n'|)}{10(2+j)} < |n-n'| \leqslant \frac{\max(|n|,|n'|)}{10(1+j)}\}} \\ &\le \|Q\|_{j,1} M^{k+j} ((k+j)!)^{\max(2,\mu)} \langle n - n' \rangle^{-(d+2)} e^{-\rho (\frac{\max(|n|,|n'|)}{10(2+j)})^{\frac{1}{\lambda}}} (1 + \max(|n|, |n'|))^{-j\tau} \\ &\le \|Q\|_{j,1} M^{k+j} ((k+j)!)^{\max(2,\mu)} \langle n - n' \rangle^{-(d+2)} e^{-\theta_0(\rho,\lambda,\tau)(1+\max(|n|,|n'|))^{\frac{1}{2\lambda}}} \\ &\le \|Q\|_{j,1} M^{N+k+j} ((k+j)!)^{\max(2,\mu)} N! \, \langle n - n' \rangle^{-(d+2)} (1 + \max(|n|, |n'|))^{-\frac{N}{2\lambda}}. \end{aligned}$$

This gives the claims in the second line of (2.37). The last claim in the proposition follows by the construction of $\widetilde{Q}$ and $\widetilde{R}$. This concludes the proof. □

We shall also need to compute the composition of three elements in $\mathcal{L}_\tau^{-j}(M,\rho,\lambda,0)$. To do that, we first have to compute the composition of one element in $\mathcal{L}_\tau^{-j}(M,\rho,\lambda,0)$ and one in $\mathcal{R}_j^{-\infty}(M,\tau)$.

Proposition 2.10. *Let $\rho > 0$, $\lambda \in [1,+\infty)$, $\tau,\tau' \in (0,1]$ and $j_1, j_2 \in \mathbb{N}$. Let $M > 1$ and $j = j_1 + j_2$. Assume $Q \in \mathcal{L}_{\tau'}^{-j_1}(M,\rho,\lambda,0)$ and $R \in \mathcal{R}_{j_2}^{-\infty}(M,\tau)$. Then*

$$\begin{aligned} Q\circ R \in \mathcal{R}_j^{-\infty}(2M,\tau), \quad R\circ Q \in \mathcal{R}_j^{-\infty}(2M,\tau),\\ |Q\circ R|_j^{(2M,\tau)} + |R\circ Q|_j^{(2M,\tau)} \lesssim \|Q\|_{j_1,0}|R|_{j_2}. \end{aligned} \tag{2.38}$$

Recall the notation $|R|_j^{(M,\tau)}$ in Definition 2.4.

Proof. We need to estimate $\|\Pi_n\partial_t^k(Q\circ R)\Pi_{n'}\|_{\mathcal{L}(L^2)}$ and $\|\Pi_n\partial_t^k(R\circ Q)\Pi_{n'}\|_{\mathcal{L}(L^2)}$ for $k\in\{0,1\}$ and for any $n,n'\in\mathbb{Z}^d$. By definition, the estimate for $\mathbf{I}''$, (2.27), for either $k=0$ or $k=1$

$$\begin{aligned} &\|\Pi_n\partial_t^k(Q\circ R)\Pi_{n'}\|_{\mathcal{L}(L^2)}\\ \le &\sum_{k_1+k_2=k}\sum_{\ell\in\mathbb{Z}^d}\binom{k}{k_1}\|\Pi_n\partial_t^{k_1}Q\Pi_\ell\|_{\mathcal{L}(L^2)}\|\Pi_\ell\partial_t^{k_2}R\Pi_{n'}\|_{\mathcal{L}(L^2)}\\ \le &\sum_{k_1+k_2=k}\sum_{\ell\in\mathbb{Z}^d}\binom{k}{k_1}\|Q\|_{j_1,0}|R|_{j_2}M^{N+j+k}[(k_1+(j_1-1)_+)!]^{\max(2,\mu)}\\ &\times[(k_2+j_2)!]^{\max(2,\mu)}N!\,e^{-\rho|\ell-n|^{\frac{1}{\lambda}}}\langle n-\ell\rangle^{-(d+2)}\langle \ell-n'\rangle^{-(d+2)}\\ &\times(1+\max(|n|,|\ell|))^{-j_1\tau'}(1+\max(|\ell|,|n'|))^{-\tau N}\mathbf{1}_{\{|\ell-n|\leqslant\frac{\max(|n|,|\ell|)}{10(1+j_1)}\}}\\ \lesssim &\|Q\|_{j_1,0}|R|_{j_2}(2M)^{N+j+k}((k+j)!)^{\max(2,\mu)}N!\\ &\times\langle n-n'\rangle^{-(d+2)}(1+\max(|n|,|n'|))^{-\tau N} \end{aligned} \tag{2.39}$$

holds for any $N\in\mathbb{N}$, any $n,n'\in\mathbb{Z}^d$, where in the last step we have used

$$(1+\max(|\ell|,|n'|))^{-\tau N}\mathbf{1}_{\{|\ell-n|\leqslant\frac{\max(|n|,|\ell|)}{10(1+j_1)}\}} \le 2^N(1+\max(|n|,|n'|))^{-\tau N}.$$

With the same reasoning, we see that the quantity after the last sign of inequality in (2.39) is also an upper bound of $\|\Pi_n\partial_t^k(R\circ Q)\Pi_{n'}\|_{\mathcal{L}(L^2)}$. Thus (2.38) holds and this concludes the proof. □

Combining Proposition 2.7 and Proposition 2.10 and remarking that $\mathcal{R}_j^{-\infty}(M,\tau)\subset\mathcal{R}_j^{-\infty}(2M,\tau)$, we obtain:

Proposition 2.11. *Let $\rho>0$, $\lambda\in[1,+\infty)$, $\tau\in(0,1]$ and $M>\theta_1(\rho,\tau)$ with θ_1 defined by (2.20). Let $j_1,j_2,j_3\in\mathbb{N}$ and $j=j_1+j_2+j_3$. Assume $Q_i\in\mathcal{L}_\tau^{-j_i}(M,\rho,\lambda,0)$, $i=1,2,3$. Then one may decompose*

$$Q_1\circ Q_2\circ Q_3 = Q+R \tag{2.40}$$

with

$$\begin{aligned} Q\in\widetilde{\mathcal{L}}_\tau^{-j}(M,\rho,\lambda,0), \quad \|Q\|_{j,0}\lesssim\|Q_1\|_{j_1,0}\|Q_2\|_{j_2,0}\|Q_3\|_{j_3,0},\\ R\in\mathcal{R}_j^{-\infty}(2M,\frac{1}{2\lambda}), \quad |R|_j^{(2M,\frac{1}{2\lambda})}\lesssim\|Q_1\|_{j_1,0}\|Q_2\|_{j_2,0}\|Q_3\|_{j_3,0}, \end{aligned} \tag{2.41}$$

where the notation $|R|_j^{(2M,\frac{1}{2\lambda})}$ is indicated in Definition 2.4.

By (2.40), its adjoint equation and (2.41) we have the following corollary which is an analogue of Corollary 2.1.

Corollary 2.3. *Under the hypotheses of Proposition 2.11, one may find self-adjoint operators* $Q \in \widetilde{\mathcal{L}}_\tau^{-j}(M, \rho, \lambda, 0)$, $R \in \mathcal{R}_j^{-\infty}(2M, \frac{1}{2\lambda})$ *such that*

$$Q_1 \circ Q_2 \circ Q_3 + (Q_1 \circ Q_2 \circ Q_3)^* = Q + R \tag{2.42}$$

with

$$\begin{aligned} \|Q\|_{j,0} &\lesssim \Big(\prod_{i=1}^{3} \|Q_i\|_{j_i,0} + \prod_{i=1}^{3} \|Q_i^*\|_{j_i,0}\Big), \\ |R|_j^{(2M,\frac{1}{2\lambda})} &\lesssim \Big(\prod_{i=1}^{3} \|Q_i\|_{j_i,0} + \prod_{i=1}^{3} \|Q_i^*\|_{j_i,0}\Big). \end{aligned} \tag{2.43}$$

3 Conjugating the equation

The goal of this section is to obtain the following: Roughly speaking, for any given $N \in \mathbb{N}^*$, we want to conjugate the operator $i\partial_t - \Delta + V$ into $i\partial_t - \Delta + V'_N + R'_N$ with V'_N exactly commuting with the modified Laplacian $\tilde{\Delta}$ and R'_N essentially being a bounded linear operator from $L^2(\mathbb{T}^d)$ to $H^N(\mathbb{T}^d)$. The process is essentially an induction. Before giving the precise description of the statement, we first present the following proposition.

Proposition 3.1. *Let $V(x, t)$ be the potential in the equation (1.1) so that it satisfies* (1.2). *Let $\tau \in (0, 1]$ and $\rho \in (0, \frac{1}{3C}]$, where the constant C is the same as in* (1.2). *Then one may find $\overline{M} > 0$ such that for any $M > \overline{M}$, the multiplication operator generated by $V(x, t)$ may be written as $Q_V + R_V$ with self-adjoint operators $Q_V \in \widetilde{\mathcal{L}}_\tau^0(M, \rho, \lambda, 0)$, $R_V \in \mathcal{R}_0^{-\infty}(M, \frac{1}{\lambda})$. Moreover,*

$$\|Q_V\|_{0,0} \le h(\lambda,\, d), \qquad |R_V|_0 \le h(\lambda,\, d), \tag{3.1}$$

where

$$h(\lambda,\, d) = C2^{\lambda d+d+2}\Big(\frac{6C\lambda(d+2)}{e}\Big)^{\lambda(d+2)}. \tag{3.2}$$

Proof. By (1.2), we know that

$$\left|\int_{\mathbb{T}^d} n^\alpha \partial_t^k V(x, t) e^{-inx} dx\right| \le (2\pi)^d C^{k+|\alpha|+1} (k!)^\mu (\alpha!)^\lambda$$

holds for any $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, any $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, any $k \in \mathbb{N}$, any $t \in \mathbb{R}$. From this inequality we deduce

$$\frac{1}{\alpha_1!} \cdots \frac{1}{\alpha_d!} \Big(\frac{|n_1|}{C}\Big)^{\frac{\alpha_1}{\lambda}} \cdots \Big(\frac{|n_d|}{C}\Big)^{\frac{\alpha_d}{\lambda}} \|\Pi_n \partial_t^k V(x, t)\|_{L^\infty}^{\frac{1}{\lambda}} \le C^{\frac{k+1}{\lambda}} (k!)^{\frac{\mu}{\lambda}}. \tag{3.3}$$

Multiplying $2^{-(\alpha_1+\cdots+\alpha_d)}$ in both sides and then taking a sum over $\alpha_1, \ldots, \alpha_d \in \mathbb{N}$, using the fact $|n_1|^{\frac{1}{\lambda}} + \cdots + |n_d|^{\frac{1}{\lambda}} \ge (|n_1| + \cdots + |n_d|)^{\frac{1}{\lambda}} \ge |n|^{\frac{1}{\lambda}}$ for $\lambda \ge 1$, we obtain after some simple calculation

$$\|\Pi_n \partial_t^k V(x, t)\|_{L^\infty} \le 2^{\lambda d} C^{k+1} (k!)^\mu e^{-\rho_0(\lambda)|n|^{\frac{1}{\lambda}}}, \tag{3.4}$$

where

$$\rho_0(\lambda) = \lambda\, (2C^{\frac{1}{\lambda}})^{-1}\,.$$

Since $\rho_0(\lambda) - \frac{1}{3C} \geq \frac{1}{6C}$ if $\lambda \geq 1$ and

$$\sup_{r\geq 1} \langle r \rangle^{d+2} \exp\{-\frac{1}{6C} r^{\frac{1}{\lambda}}\} \leq 2^{d+2} \Big(\frac{6C\lambda(d+2)}{e}\Big)^{\lambda(d+2)},$$

we have

$$C2^{\lambda d} \exp\{-\rho_0(\lambda)|n|^{\frac{1}{\lambda}}\} \leq h(\lambda, d) \exp\{-\frac{1}{3C}|n|^{\frac{1}{\lambda}}\} \langle n \rangle^{-(d+2)},$$

where $h(\lambda, d)$ is given by (3.2). Thus by (3.4),

$$\|\Pi_n \partial_t^k V(x,t)\|_{L^\infty} \leq h(\lambda, d) C^k (k!)^\mu e^{-\frac{1}{3C}|n|^{\frac{1}{\lambda}}} \langle n \rangle^{-(d+2)}.$$

Therefore if $\rho \in (0, \frac{1}{3C}]$, then

$$\|\Pi_n \partial_t^k V \Pi_{n'}\|_{\mathcal{L}(L^2)} \leq \|\Pi_{n-n'} \partial_t^k V(x,t)\|_{L^\infty} \leq h(\lambda, d) C^k (k!)^\mu \, e^{-\rho|n-n'|^{\frac{1}{\lambda}}} \langle n - n' \rangle^{-(d+2)}.$$

We define

$$Q_V = \sum_{n\in\mathbb{Z}^d} \sum_{n'\in\mathbb{Z}^d} \Pi_n V \Pi_{n'} \mathbf{1}_{\{|n-n'| \leqslant \frac{\max(|n|,|n'|)}{20}\}},$$
$$R_V = \sum_{n\in\mathbb{Z}^d} \sum_{n'\in\mathbb{Z}^d} \Pi_n V \Pi_{n'} \mathbf{1}_{\{|n-n'| > \frac{\max(|n|,|n'|)}{20}\}}.$$

By the above formulas, for any $M \geq C$, we have $Q_V \in \widetilde{\mathcal{L}}_\tau^0(M, \rho, \lambda, 0)$ with $\|Q_V\|_{0,0} \leq h(\lambda,\, d)$. For $k \in \{0, 1\}$, we know that

$$\begin{aligned}\|\Pi_n \partial_t^k R_V \Pi_{n'}\|_{\mathcal{L}(L^2)} &\leq h(\lambda,\, d) C^k e^{-\frac{1}{3C}\,|n-n'|^{\frac{1}{\lambda}}} \langle n - n' \rangle^{-(d+2)} \mathbf{1}_{\{\,|n-n'| > \frac{\max(|n|,|n'|)}{20}\}} \\ &\leq h(\lambda,\, d) C^k (120C)^N N! \langle n - n' \rangle^{-(d+2)} \Big(1 + \max(|n|, |n'|)\Big)^{-\frac{N}{\lambda}}\end{aligned}$$

holds for any $N \in \mathbb{N}$, where we have used

$$\max(|n|, |n'|) \mathbf{1}_{\{|n-n'| > \frac{\max(|n|,|n'|)}{20}\}} \geq \frac{1}{2}(1 + \max(|n|, |n'|)).$$

If $M > \overline{M} \overset{def}{=} 120C$, then $R_V \in \mathcal{R}_0^{-\infty}(M, \frac{1}{\lambda})$ and $|R_V|_0 \leq h(\lambda,\, d)$. This concludes the proof. $\square$

Remark 3.1. *Let $\sigma \in (0, \frac{1}{10})$ and $\tau_0 \in (0, \sigma)$ be given by Proposition 2.1. From now on, we fix $\tau = \min(\frac{\tau_0}{\lambda}, \frac{1}{2\lambda}) = \frac{\tau_0}{\lambda}$ and fix $\rho \in (0, \frac{1}{3C}]$. We also fix $M > \max(\overline{M},\ 2\theta_1(\rho, \tau)) \geq \frac{2}{\rho}$ so that all the conclusions in Section 2 and Proposition 3.1 hold, where $\theta_1(\rho, \tau)$ is given by (2.20). We choose those M, τ because they will be used in the argument of the following proposition. Note that M depends on λ, but this dependence does not matter in the sequel.*

The main result of this section is the following:

Proposition 3.2. *Let $m \in \mathbb{N}^*$ and denote $P_0 = i\partial_t - \Delta$. Let K be a large constant. There are sequences $(Q'_j)_{1\leq j\leq m}$, $(Q''_j)_{1\leq j\leq m}$ satisfying*

$$Q'_j \in \mathcal{L}_\tau^{-j}(M, \rho, \lambda, 0), \quad Q'^*_j = -Q'_j, \quad \|Q'_j\|_{j,0} \leq \frac{K^{j-\frac{1}{2}}}{j^2} h(\lambda, d)^j; \tag{3.5}$$

$$[Q'_j, \Delta] \in \mathcal{L}_\tau^{-(j-1)}(M, \rho, \lambda, 0), \qquad \|[Q'_j, \Delta]\|_{j-1,0} \leq \frac{K^{j-1}}{j^2} h(\lambda, d)^j; \tag{3.6}$$

$$Q''_j \in \mathcal{L}_\tau^{-(j+1)}(M, \rho, \lambda, 0), \quad Q''^*_j = Q''_j, \quad \|Q''_j\|_{j+1,0} \leq \frac{K^{j+\frac{1}{2}}}{(j+1)^2} h(\lambda, d)^{j+1}; \tag{3.7}$$

$$[Q''_j, \Delta] \in \mathcal{L}_\tau^{-j}(M, \rho, \lambda, 0), \qquad \|[Q''_j, \Delta]\|_{j,0} \leq \frac{K^j}{(j+1)^2} h(\lambda, d)^{j+1} \tag{3.8}$$

such that if we set $Q_j = Q'_j + Q''_j$, $Q^m = \sum_{j=1}^m Q_j$

$$
\begin{aligned}
&(I+Q^m)^*(P_0+V)(I+Q^m)\\
&\quad = i\partial_t - \Delta + V^m + \frac{1}{2}\sum_{j=m+1}^{2m+1}(S_jP_0+P_0S_j) + \frac{1}{2}\sum_{j=1}^{2m+1}(R_jP_0+P_0R_j)\\
&\qquad + \widetilde{S}_{m+1} + \sum_{j=m+1}^{2m+3}\overline{S}_j + \sum_{j=2}^{2m+3}\overline{R}_j + \sum_{j=1}^{m}\widehat{R}_j
\end{aligned}
\tag{3.9}
$$

where the terms in the right hand side of (3.9) satisfy the following conditions:

- V^m, S_j, R_j, $\widetilde{S}_j$, $\overline{S}_j$, $\overline{R}_j$, $\widehat{R}_j$ *are self-adjoint;*
- $[V^m, \tilde{\Delta}] = 0$;
- $S_j \in \mathcal{L}_\tau^{-(j+1)}(M,\rho,\lambda,0)$, $\quad \|S_j\|_{j+1,0} \lesssim \frac{K^j}{(j+1)^2}h(\lambda,d)^{j+1}$, $\quad m+1 \le j \le 2m+1$;
- $[\Delta, S_j] \in \widetilde{\mathcal{L}}_\tau^{-j}(M,\rho,\lambda,0)$, $\quad \|[\Delta,S_j]\|_{j,0} \lesssim \frac{K^{j-\frac{1}{2}}}{(j+1)^2}h(\lambda,d)^{j+1}$, $\quad m+1 \le j \le 2m+1$
- $R_j \in \mathcal{R}_{j+1}^{-\infty}(M,\tau)$, $\quad |R_j|_{j+1} \lesssim \frac{K^j}{(j+1)^2}h(\lambda,d)^{j+1}$, $\quad 1 \le j \le 2m+1$;
- $\widetilde{S}_{m+1} \in \widetilde{\mathcal{L}}_\tau^{-m}(M,\rho,\lambda,1)$, $\quad \|\widetilde{S}_{m+1}\|_{m,1} \lesssim \frac{K^{m-\frac{1}{2}}}{(m+1)^2}h(\lambda,d)^m$;
- $\overline{S}_j \in \widetilde{\mathcal{L}}_\tau^{-(j-1)}(M,\rho,\lambda,0)$, $\quad \|\overline{S}_j\|_{j-1,0} \lesssim \frac{K^{j-\frac{3}{2}}}{j^2}h(\lambda,d)^j$, $\quad m+1 \le j \le 2m+3$;
- $\overline{R}_j \in \mathcal{R}_{j-1}^{-\infty}(4M,\tau)$, $\quad |\overline{R}|_{j-1}^{(4M,\tau)} \lesssim \frac{K^{j-\frac{3}{2}}}{j^2}h(\lambda,d)^j$, $\quad 2 \le j \le 2m+3$;
- $\widehat{R}_j \in \mathcal{R}_{j-1}^{-\infty}(M,\tau)$, $\quad |\widehat{R}_j|_{j-1} \lesssim \frac{K^{j-\frac{3}{2}}}{j^2}h(\lambda,d)^j$, $\quad 1 \le j \le m$.

The notation $|\overline{R}|_{j-1}^{(4M,\tau)}$ *is explained in Definition 2.4 and by* $A \lesssim B$ *we mean that there is an absolute constant* C *such that* $A \le CB$.

Let us first compute the left hand side of (3.9).

Lemma 3.1. *Let* Q'_j, Q''_j *be given operators satisfying (3.5)– (3.8) for* $1 \le j \le m$. *Denote* $Q'^m = \sum_{j=1}^m Q'_j$, $Q''^m = \sum_{j=1}^m Q''_j$. *Then one may find*

- *Elements* $(S_j)_{1\le j\le 2m+1}$, $(R_j)_{1\le j\le 2m+1}$ *satisfying*
  - (1) $S_j \in \widetilde{\mathcal{L}}_\tau^{-(j+1)}(M,\rho,\lambda,0)$, $\quad \|S_j\|_{j+1,0} \lesssim \frac{K^j}{(j+1)^2}h(\lambda,d)^{j+1}$, $1 \le j \le 2m+1$;
  - (2) $[\Delta, S_j] \in \widetilde{\mathcal{L}}_\tau^{-j}(M,\rho,\lambda,0)$, $\|[\Delta,S_j]\|_{j,0} \lesssim \frac{K^{j-\frac{1}{2}}}{(j+1)^2}h(\lambda,d)^{j+1}$, $1 \le j \le 2m+1$;
  - (3) $R_j \in \mathcal{R}_{j+1}^{-\infty}(M,\tau)$, $\quad |R_j|_{j+1} \lesssim \frac{K^j}{(j+1)^2}h(\lambda,d)^{j+1}$, $\quad 1 \le j \le 2m+1$;
  - (4) S_j, R_j *are self-adjoint and depend only on* Q'_ℓ, $1 \le \ell \le \min(j,m)$, Q''_ℓ, $1 \le \ell < \min(j,m+1)$;
- *Elements* $(\widetilde{S}_j)_{2\le j\le m+1}$, $(\overline{S}_j)_{2\le j\le 2m+3}$, $(\overline{R}_j)_{2\le j\le 2m+3}$ *satisfying*
  - (5) $\widetilde{S}_j \in \widetilde{\mathcal{L}}_\tau^{-(j-1)}(M,\rho,\lambda,1)$, $\quad \|\widetilde{S}_j\|_{j-1,1} \lesssim \frac{K^{j-\frac{3}{2}}}{j^2}h(\lambda,d)^{j-1}$, $\quad 2 \le j \le m+1$;

(6) $\overline{S}_j \in \widetilde{\mathcal{L}}_\tau^{-(j-1)}(M,\rho,\lambda,0)$, $\quad \|\overline{S}_j\|_{j-1,0} \lesssim \frac{K^{j-\frac{3}{2}}}{j^2} h(\lambda,d)^j, \quad 2 \le j \le 2m+3;$

(7) $\overline{R}_j \in \mathcal{R}_{j-1}^{-\infty}(4M,\tau)$, $\qquad |\overline{R}_j|_{j-1}^{(4M,\tau)} \lesssim \frac{K^{j-\frac{3}{2}}}{j^2} h(\lambda,d)^j, \quad 2 \le j \le 2m+3;$

(8) $\widetilde{S}_j$, $\overline{S}_j$, $\overline{R}_j$ *are self-adjoint and depend only on* Q'_ℓ, Q''_ℓ, $1 \le \ell < \min(j, m+1)$,

such that

$$
\begin{aligned}
(I+Q^m)^*(P_0+V)(I+Q^m) & \\
&= i\partial_t - \Delta + V + [Q'^m, \Delta] + Q''^m P_0 + P_0 Q''^m \\
&\quad + \frac{1}{2}\sum_{j=1}^{2m+1} \left(S_j P_0 + P_0 S_j\right) + \frac{1}{2}\sum_{j=1}^{2m+1}\left(R_j P_0 + P_0 R_j\right) \\
&\quad + \sum_{j=2}^{m+1} \widetilde{S}_j + \sum_{j=2}^{2m+3} \overline{S}_j + \sum_{j=2}^{2m+3} \overline{R}_j.
\end{aligned} \tag{3.10}
$$

Proof of Lemma 3.1: Using that $(Q'^m)^* = -Q'^m$, $(Q''^m)^* = Q''^m$, we write

$$
\begin{aligned}
(I+Q^m)^*(P_0+V)(I+Q^m) &= i\partial_t - \Delta + V \\
&\quad + [Q'^m, \Delta] - [Q'^m, i\partial_t] + Q''^m P_0 + P_0 Q''^m && (3.11)\\
&\quad + \frac{1}{2}\Big((Q^m)^* Q^m P_0 + P_0 (Q^m)^* Q^m\Big) && (3.12)\\
&\quad + \frac{1}{2}\Big((Q^m)^*[i\partial_t, Q^m] + [(Q^m)^*, i\partial_t] Q^m\Big) && (3.13)\\
&\quad + \frac{1}{2}\Big((Q^m)^*[-\Delta, Q^m] + [(Q^m)^*, -\Delta] Q^m\Big) && (3.14)\\
&\quad + (Q^m)^* V + V Q^m + (Q^m)^* V Q^m. && (3.15)
\end{aligned}
$$

Let us show how the right hand side contributes to that of (3.10). We deal with it term by term.

We write by Corollary 2.1 and Notation 4

$$
\begin{aligned}
(Q^m)^* Q^m =& \frac{1}{2}\sum_{j=1}^{2m-1} \sum_{\substack{j_1+j_2=j+1\\ 1\le j_1,j_2\le m}} \mathscr{M}'(Q'^*_{j_1}, Q'_{j_2}) + \mathscr{R}'(Q'^*_{j_1}, Q'_{j_2}) \\
&+ \sum_{j=2}^{2m} \sum_{\substack{j_1+j_2=j\\ 1\le j_1,j_2\le m}} \mathscr{M}'(Q''^*_{j_1}, Q'_{j_2}) + \mathscr{R}'(Q''^*_{j_1}, Q'_{j_2}) \\
&+ \frac{1}{2}\sum_{j=3}^{2m+1} \sum_{\substack{j_1+j_2=j-1\\ 1\le j_1,j_2\le m}} \mathscr{M}'(Q''^*_{j_1}, Q''_{j_2}) + \mathscr{R}'(Q''^*_{j_1}, Q''_{j_2}) \\
=& \sum_{j=1}^{2m-1} (S_j^{(1)} + R_j^{(1)}) + \sum_{j=2}^{2m} (S_j^{(2)} + R_j^{(2)}) + \sum_{j=3}^{2m+1} (S_j^{(3)} + R_j^{(3)})
\end{aligned} \tag{3.16}
$$

for self-adjoint operators $S_j^{(i)} \in \widetilde{\mathcal{L}}_\tau^{-(j+1)}(M,\rho,\lambda,0)$, $R_j^{(i)} \in \mathcal{R}_{j+1}^{\infty}(M, \frac{1}{2\lambda}) \subset \mathcal{R}_{j+1}^{\infty}(M,\tau)$, $i = 1,2,3$, $j = 1,\dots,2m+1$. We make the following convention: **we set the terms that do not appear to be zero.** For instance, here we set

$$
\begin{aligned}
S_j^{(1)} &= R_j^{(1)} = 0, \quad j = 2m, 2m+1, \\
S_j^{(2)} &= R_j^{(2)} = 0, \quad j = 1, 2m+1 \\
S_j^{(3)} &= R_j^{(3)} = 0, \quad j = 1, 2.
\end{aligned}
$$

We shall use such a convention throughout the proof of Lemma 3.1. Using (2.34), (3.5), (3.7) and the fact that

$$\sum_{\substack{j_1+j_2=j+1\\1\le j_1,j_2\le m}}\frac{1}{j_1^2}\cdot\frac{1}{j_2^2}+2\sum_{\substack{j_1+j_2=j\\1\le j_1,j_2\le m}}\frac{1}{j_1^2}\cdot\frac{1}{j_2^2}+\sum_{\substack{j_1+j_2=j-1\\1\le j_1,j_2\le m}}\frac{1}{j_1^2}\cdot\frac{1}{j_2^2}\lesssim\frac{1}{(j+1)^2},\tag{3.17}$$

we obtain

$$\sum_{i=1}^{3}\left(\|S_j^{(i)}\|_{j+1,0}+|R_j^{(i)}|_{j+1}\right)\lesssim\frac{K^j}{(j+1)^2}h(\lambda,d)^{j+1},\quad 1\le j\le 2m+1.$$

Defining

$$S_j=\sum_{i=1}^{3}S_j^{(i)},\quad 1\le j\le 2m+1;\qquad R_j=\sum_{i=1}^{3}R_j^{(i)},\quad 1\le j\le 2m+1,$$

we know by the construction that S_j, R_j satisfy (1), (3) and (4). Moreover, by expressions (2.16) and (2.17), we get

$$\begin{aligned}[\Delta,S_j]&=\sum_{i=1}^{3}[\Delta,S_j^{(i)}]\\&=\sum_{\substack{j_1+j_2=j+1\\1\le j_1,j_2\le m}}[\mathscr{M}([\Delta,Q_{j_1}'^{*}],Q_{j_2}')+\mathscr{M}(Q_{j_1}'^{*},[\Delta,Q_{j_2}'])]\\&\quad+\sum_{\substack{j_1+j_2=j\\1\le j_1,j_2\le m}}\Big[\mathscr{M}([\Delta,Q_{j_1}'^{*}],Q_{j_2}'')+\mathscr{M}(Q_{j_1}'^{*},[\Delta,Q_{j_2}''])\\&\qquad\qquad+\mathscr{M}([\Delta,Q_{j_1}''^{*}],Q_{j_2}')+\mathscr{M}(Q_{j_1}''^{*},[\Delta,Q_{j_2}'])\Big]\\&\quad+\sum_{\substack{j_1+j_2=j-1\\1\le j_1,j_2\le m}}[\mathscr{M}([\Delta,Q_{j_1}''^{*}],Q_{j_2}'')+\mathscr{M}(Q_{j_1}''^{*},[\Delta,Q_{j_2}''])],\end{aligned}$$

so we know from (3.5)–(3.8), Proposition 2.7 and (3.17) that $[\Delta,S_j]\in\widetilde{\mathcal{L}}_\tau^{-j}(M,\rho,\lambda,0)$ and $|||[\Delta,S_j]|||_{j,0}\lesssim\frac{K^{j-\frac12}}{(j+1)^2}h(\lambda,d)^{j+1}$. Therefore, (3.12) contributes to the third line of (3.10).

By Proposition 2.4, Proposition 2.9, one may decompose

$$-[Q_{j-1}',i\partial_t]=\widetilde{S}_j+\widetilde{R}_j,\quad 2\le j\le m+1\tag{3.18}$$

with

$$\begin{aligned}&\widetilde{S}_j\in\widetilde{\mathcal{L}}_\tau^{-(j-1)}(M,\rho,\lambda,1),\quad \|\widetilde{S}_j\|_{j-1,1}\le\frac{K^{j-\frac32}}{(j-1)^2}h(\lambda,d)^{j-1},\ 2\le j\le m+1;\\&\widetilde{R}_j\in\mathcal{R}_{j-1}^{\infty}(M,\frac{1}{2\lambda})\subset\mathcal{R}_{j-1}^{-\infty}(M,\tau),\quad |\widetilde{R}_j|_{j-1}\le\frac{K^{j-\frac32}}{(j-1)^2}h(\lambda,d)^{j-1},\quad 2\le j\le m+1.\end{aligned}\tag{3.19}$$

Since $-[Q_{j-1}',i\partial_t]$ is self-adjoint, so are $\widetilde{S}_j$ and $\widetilde{R}_j$. Thus this determines the first term in the fourth line of (3.10) and $\widetilde{R}_j$ contributes to $\overline{R}_j$.

According to Proposition 2.4, Corollary 2.2 and Notation 4, we may write

$$
\begin{aligned}
(3.13) = &\frac{1}{2}\sum_{j=3}^{2m+1}\sum_{\substack{j_1+j_2=j-1\\1\le j_1,j_2\le m}} \mathscr{M}'(Q'^*_{j_1},[i\partial_t,Q'_{j_2}])+\mathscr{R}'(Q'^*_{j_1},[i\partial_t,Q'_{j_2}]),\\
&+\frac{1}{2}\sum_{j=4}^{2m+2}\sum_{\substack{j_1+j_2=j-2\\1\le j_1,j_2\le m}} \mathscr{M}'(Q''^*_{j_1},[i\partial_t,Q'_{j_2}])+\mathscr{R}'(Q''^*_{j_1},[i\partial_t,Q'_{j_2}])\\
&+\frac{1}{2}\sum_{j=4}^{2m+2}\sum_{\substack{j_1+j_2=j-2\\1\le j_1,j_2\le m}} \mathscr{M}'(Q'^*_{j_1},[i\partial_t,Q''_{j_2}])+\mathscr{R}'(Q'^*_{j_1},[i\partial_t,Q''_{j_2}]) \qquad (3.20)\\
&+\frac{1}{2}\sum_{j=5}^{2m+3}\sum_{\substack{j_1+j_2=j-3\\1\le j_1,j_2\le m}} \mathscr{M}'(Q''^*_{j_1},[i\partial_t,Q''_{j_2}])+\mathscr{R}'(Q''^*_{j_1},[i\partial_t,Q''_{j_2}])\\
=&\sum_{j=3}^{2m+1}(\overline{S}^{(1)}_j+\overline{R}^{(1)}_j)+\sum_{j=4}^{2m+2}(\overline{S}^{(2)}_j+\overline{R}^{(2)}_j)+\sum_{j=5}^{2m+3}(\overline{S}^{(3)}_j+\overline{R}^{(3)}_j)
\end{aligned}
$$

for self-adjoint operators $\overline{S}^{(i)}_j \in \widetilde{\mathcal{L}}_\tau^{-(j-1)}(M,\rho,\lambda,0)$, $\overline{R}^{(i)}_j \in \mathcal{R}^{\infty}_{j-1}(M,\frac{1}{2\lambda}) \subset \mathcal{R}^{-\infty}_{j-1}(M,\tau)$, $1 \le i \le 3$, $3 \le j \le 2m+3$. Here we have used the convention made on page 17. By the inequalities which are contained in the statement of Corollary 2.2, (2.13), (3.5), (3.7) and (3.17), we obtain

$$
\sum_{i=1}^{3}\left(\|\overline{S}^{(i)}_j\|_{j-1,0}+|\overline{R}^{(i)}_j|_{j-1}\right) \lesssim \frac{K^{j-2}}{j^2}h(\lambda,d)^{j-1}, \quad 3 \le j \le 2m+3. \qquad (3.21)
$$

Now we turn to the term (3.14). Using Notation 4, Corollary 2.1, (3.5)–(3.8), we write

$$
\begin{aligned}
(3.14) = &\frac{1}{2}\sum_{j=2}^{2m}\sum_{\substack{j_1+j_2=j\\1\le j_1,j_2\le m}} \mathscr{M}'(Q'^*_{j_1},[-\Delta,Q'_{j_2}])+\mathscr{R}'(Q'^*_{j_1},[-\Delta,Q'_{j_2}])\\
&+\frac{1}{2}\sum_{j=3}^{2m+1}\sum_{\substack{j_1+j_2=j-1\\1\le j_1,j_2\le m}} \mathscr{M}'(Q''^*_{j_1},[-\Delta,Q'_{j_2}])+\mathscr{R}'(Q''^*_{j_1},[-\Delta,Q'_{j_2}])\\
&+\frac{1}{2}\sum_{j=3}^{2m+1}\sum_{\substack{j_1+j_2=j-1\\1\le j_1,j_2\le m}} \mathscr{M}'(Q'^*_{j_1},[-\Delta,Q''_{j_2}])+\mathscr{R}'(Q'^*_{j_1},[-\Delta,Q''_{j_2}])\\
&+\frac{1}{2}\sum_{j=4}^{2m+2}\sum_{\substack{j_1+j_2=j-2\\1\le j_1,j_2\le m}} \mathscr{M}'(Q''^*_{j_1},[-\Delta,Q''_{j_2}])+\mathscr{R}'(Q''^*_{j_1},[-\Delta,Q''_{j_2}])\\
=&\sum_{j=2}^{2m}\left(\overline{S}^{(4)}_j+\overline{R}^{(4)}_j\right)+\sum_{j=3}^{2m+1}\left(\overline{S}^{(5)}_j+\overline{R}^{(5)}_j\right)+\sum_{j=4}^{2m+2}\left(\overline{S}^{(6)}_j+\overline{R}^{(6)}_j\right)
\end{aligned}
$$

for self-adjoint operators $\overline{S}^{(i)}_j \in \widetilde{\mathcal{L}}_\tau^{-(j-1)}(M,\rho,\lambda,0)$, $\overline{R}^{(i)}_j \in \mathcal{R}^{\infty}_{j-1}(M,\frac{1}{2\lambda}) \subset \mathcal{R}^{-\infty}_{j-1}(M,\tau)$, $4 \le i \le 6$, $2 \le j \le 2m+2$, using the convention made on page 17. By (2.34), (3.5)– (3.8) and (3.17) we have

$$
\sum_{i=4}^{6}\left(\|\overline{S}^{(i)}_j\|_{j-1,0}+|\overline{R}^{(i)}_j|_{j-1}\right) \lesssim \frac{K^{j-\frac{3}{2}}}{j^2}h(\lambda,d)^{j}, \quad 2 \le j \le 2m+2.
$$

Let us now analyze (3.15). By Proposition 3.1, Corollary 2.1 and Proposition 2.10, we write

$$
\begin{aligned}
(Q^m)^*V + VQ^m &= \sum_{j=2}^{m+1} [Q'^*_{j-1}(Q_V + R_V) + (Q_V + R_V)Q'_{j-1}] \\
&\quad + \sum_{j=3}^{m+2} [(Q''_{j-2})^*(Q_V + R_V) + (Q_V + R_V)Q''_{j-2}] \\
&= \sum_{j=2}^{m+1} \left(\overline{S}_j^{(7)} + \overline{R}_j^{(7)}\right) + \sum_{j=3}^{m+2} \left(\overline{S}_j^{(8)} + \overline{R}_j^{(8)}\right)
\end{aligned}
$$

for self-adjoint operators $\overline{S}_j^{(i)} \in \widetilde{\mathcal{L}}_\tau^{-(j-1)}(M, \rho, \lambda, 0)$, $\overline{R}_j^{(i)} \in \mathcal{R}_{j-1}^{\infty}(2M, \frac{1}{2\lambda}) \subset \mathcal{R}_{j-1}^{-\infty}(2M, \tau)$, $7 \le i \le 8$, $2 \le j \le m+2$. In this case the convention reads that

$$\overline{S}_{m+2}^{(7)} = \overline{R}_{m+2}^{(7)} = \overline{S}_2^{(8)} = \overline{R}_2^{(8)} = 0.$$

Moreover, by (2.34), (3.1), (3.5), (3.7) and (3.17)

$$\sum_{i=7}^{8} \left(\|\overline{S}_j^{(i)}\|_{j-1,0} + |\overline{R}_j^{(i)}|_{j-1}^{(2M,\tau)}\right) \lesssim \frac{K^{j-\frac{3}{2}}}{j^2} h(\lambda, d)^j, \quad 2 \le j \le m+2. \tag{3.22}$$

Similarly, by Proposition 3.1, Proposition 2.10, Corollary 2.3, we also have

$$
\begin{aligned}
(Q^m)^*VQ^m =& \frac{1}{2} \sum_{j=3}^{2m+1} \sum_{\substack{j_1+j_2=j-1 \\ 1 \le j_1, j_2 \le m}} Q'^*_{j_1}(Q_V + R_V)Q'_{j_2} + Q'^*_{j_2}(Q_V + R_V)Q'_{j_1} \\
&+ \sum_{j=4}^{2m+2} \sum_{\substack{j_1+j_2=j-2 \\ 1 \le j_1, j_2 \le m}} Q''^*_{j_1}(Q_V + R_V)Q'_{j_2} + Q'^*_{j_2}(Q_V + R_V)Q''_{j_1} \\
&+ \frac{1}{2} \sum_{j=5}^{2m+3} \sum_{\substack{j_1+j_2=j-3 \\ 1 \le j_1, j_2 \le m}} Q''^*_{j_1}(Q_V + R_V)Q''_{j_2} + Q''^*_{j_2}(Q_V + R_V)Q''_{j_1} \\
=& \sum_{j=3}^{2m+1} \left(\overline{S}_j^{(9)} + \overline{R}_j^{(9)}\right) + \sum_{j=4}^{2m+2} \left(\overline{S}_j^{(10)} + \overline{R}_j^{(10)}\right) + \sum_{j=5}^{2m+3} \left(\overline{S}_j^{(11)} + \overline{R}_j^{(11)}\right)
\end{aligned}
$$

for self-adjoint operators $\overline{S}_j^{(i)} \in \widetilde{\mathcal{L}}_\tau^{-(j-1)}(M, \rho, \lambda, 0)$, $\overline{R}_j^{(i)} \in \mathcal{R}_{j-1}^{\infty}(4M, \frac{1}{2\lambda}) \subset \mathcal{R}_{j-1}^{-\infty}(4M, \tau)$, $9 \le i \le 11$, $3 \le j \le 2m+3$ and by (2.43), (2.38), (3.5), (3.7) and (3.17)

$$\sum_{i=9}^{11} \left(\|\overline{S}_j^{(i)}\|_{j-1,0} + |\overline{R}_j^{(i)}|_{j-1}^{(4M,\tau)}\right) \lesssim \frac{K^{j-2}}{j^2} h(\lambda, d)^j, \quad 3 \le j \le 2m+3. \tag{3.23}$$

Using the convention made on page 17, we set

$$\overline{S}_j = \sum_{i=1}^{11} \overline{S}_j^{(i)}, \quad \overline{R}_j = \widetilde{R}_j + \sum_{i=1}^{11} \overline{R}_j^{(i)}, \quad 2 \le j \le 2m+3.$$

Since $\mathcal{R}_{j-1}^{-\infty}(M, \tau) \subset \mathcal{R}_{j-1}^{-\infty}(4M, \tau)$, we see from (3.19) to (3.23) that $(\overline{S}_j)_{2 \le j \le 2m+3}$, $(\overline{R}_j)_{2 \le j \le 2m+3}$ satisfy the conditions listed in the lemma and contribute respectively to the second and last terms in the last line of (3.10). This concludes the proof. □

Proof of Proposition 3.2: We shall recursively construct $Q'_1, Q''_1, \dots, Q'_m, Q''_m$ with the required estimates so that the left hand side of (3.9) may be written for $r = 1, \dots, m+1$

$$\begin{aligned} i\partial_t - \Delta + V^{r-1} &+ \sum_{j=r}^{m}[Q'_j, \Delta] + \sum_{j=r}^{m}(Q''_j P_0 + P_0 Q''_j) \\ &+ \frac{1}{2}\sum_{j=r}^{2m+1}(S_j P_0 + P_0 S_j) + \frac{1}{2}\sum_{j=1}^{2m+1}(R_j P_0 + P_0 R_j) \\ &+ \sum_{j=r}^{m+1}\widetilde{S}_j + \sum_{j=r}^{2m+3}\overline{S}_j + \sum_{j=1}^{2m+3}\overline{R}_j + \sum_{j=1}^{r-1}\widehat{R}_j, \end{aligned} \tag{3.24}$$

where $V^0 = 0$, $(V^j)^* = V^j$ and $[\tilde{\Delta}, V^j] = 0$ for $j \geq 1$, $\widetilde{S}_1 = 0$, $\overline{S}_1 = Q_V$, $\overline{R}_1 = R_V$. Here Q_V, R_V are defined in Proposition 3.1. Remark that without regard to all the estimates, (3.24) with $r = 1$ is the conclusion of Lemma 3.1 and (3.24) with $r = m+1$ is the conclusion we want to reach. Assume that (3.24) has been obtained at rank r and we have already had the estimates (3.5)–(3.8) for $Q'_1, \dots, Q'_{r-1}$, $Q''_1, \dots, Q''_{r-1}$. By Lemma 3.1, we have determined S_ℓ, R_ℓ, $1 \leq \ell \leq r-1$, $\widetilde{S}_\ell, \overline{S}_\ell, \overline{R}_\ell$, $1 \leq \ell \leq r$ and they also satisfy the estimates listed in Lemma 3.1. Using Notation 3, we set $V^r = V^{r-1} + (\widetilde{S}_r)_D + (\overline{S}_r)_D$ and denote

$$\begin{aligned} (\widetilde{S}_r)^M_{ND} &= \sum_{n,n' \in \mathbb{Z}^d} \Pi_n(\widetilde{S}_r)_{ND}\Pi_{n'}\mathbf{1}_{\{||n|^2-|n'|^2| > \frac{1}{4}(|n|+|n'|)^{\tau_0}\}}, \\ (\overline{S}_r)^M_{ND} &= \sum_{n,n' \in \mathbb{Z}^d} \Pi_n(\overline{S}_r)_{ND}\Pi_{n'}\mathbf{1}_{\{||n|^2-|n'|^2| > \frac{1}{4}(|n|+|n'|)^{\tau_0}\}}, \end{aligned} \tag{3.25}$$

with τ_0 given by Proposition 2.1. We now deduce from (2.10) and Proposition 2.2 that $[\tilde{\Delta}, V^r] = 0$, $(\widetilde{S}_r)^M_{ND} \in \overline{\mathcal{L}}^{-(r-1)}_{\tau,ND}(M,\rho,\lambda,1)$ and $(\overline{S}_r)^M_{ND} \in \overline{\mathcal{L}}^{-(r-1)}_{\tau,ND}(M,\rho,\lambda,0)$. We let Q'_r satisfy

$$[Q'_r, \Delta] = -(\widetilde{S}_r)^M_{ND} - (\overline{S}_r)^M_{ND}. \tag{3.26}$$

Since $\tau_0 \geq \tau$ by Remark 3.1, according to Proposition 2.3 this equation defines an element $Q'_r \in \mathcal{L}^{-r}_\tau(M,\rho,\lambda,0)$ with

$$\|Q'_r\|_{r,0} \lesssim \|(\widetilde{S}_r)^M_{ND}\|_{r-1,1} + \|(\overline{S}_r)^M_{ND}\|_{r-1,0} \lesssim \frac{K^{r-\frac{3}{2}}}{r^2}h(\lambda,d)^r \leq \frac{K^{r-\frac{1}{2}}}{r^2}h(\lambda,d)^r, \tag{3.27}$$

if K is larger than the implicit constant and since $(\widetilde{S}_r)^M_{ND}$, $(\overline{S}_r)^M_{ND}$ are self-adjoint, $Q'^*_r = -Q'_r$. (3.6) with $j = r$ follows from (3.26), (5) and (6) with $j = r$ if K is larger than the square of the implicit constant. Thus Q'_r satisfies (3.5) and (3.6). We then claim that $(\widetilde{S}_r)_{ND} - (\widetilde{S}_r)^M_{ND}$ and $(\overline{S}_r)_{ND} - (\overline{S}_r)^M_{ND}$ contribute to $\widehat{R}_r$. But

$$\Pi_n((\widetilde{S}_r)_{ND} - (\widetilde{S}_r)^M_{ND})\Pi_{n'} = \Pi_n(\widetilde{S}_r)_{ND}\Pi_{n'}\mathbf{1}_{\{||n|^2-|n'|^2| \leq \frac{1}{4}(|n|+|n'|)^{\tau_0}\}} \tag{3.28}$$

and since $(\widetilde{S}_r)_{ND} \in \widetilde{\mathcal{L}}^{-(r-1)}_{\tau,ND}(M,\rho,\lambda,1)$, this expression is non zero only when n and n' belong to A_α and A_β with $\alpha \neq \beta$, where A_α and A_β are defined in Proposition 2.1. So the second condition in Proposition 2.1, together with the cut-off in (3.28), implies that $|n - n'| \geq \frac{1}{2}(1 + \max(|n|,|n'|))^{\tau_0}$. Then it follows by (2.8) and the assumption $M > \frac{2}{\rho}$ stated in Remark 3.1 that

$$\begin{aligned} &\|\Pi_n\partial_t^k((\widetilde{S}_r)_{ND} - (\widetilde{S}_r)^M_{ND})\Pi_{n'}\|_{\mathcal{L}(L^2)} \\ &\quad\lesssim \|(\widetilde{S}_r)_{ND}\|_{r-1,1}M^{N+k+r-1}[(k+r-1)!]^{\max(2,\mu)}N! \\ &\quad\quad\times \langle n-n'\rangle^{-(d+2)}(1+\max(|n|,|n'|))^{-\frac{\tau_0 N}{\lambda}} \end{aligned} \tag{3.29}$$

for any $k \in \{0,1\}$, any $N \in \mathbb{N}$, any $n, n' \in \mathbb{Z}^d$. With the same reasoning we can get a similar estimate for $\|\Pi_n((\overline{S}_r)_{ND} - (\overline{S}_r)^M_{ND})\Pi_{n'}\|_{\mathcal{L}(L^2)}$. We then set

$$\widehat{R}_r = (\widetilde{S}_r)_{ND} - (\widetilde{S}_r)^M_{ND} + (\overline{S}_r)_{ND} - (\overline{S}_r)^M_{ND}$$

and deduce from (3.29), a similar estimate to (3.29) for $\|\Pi_n((\overline{S}_r)_{ND} - (\overline{S}_r)^M_{ND})\Pi_{n'}\|_{\mathcal{L}(L^2)}$, (2.10), (5) and (6) with $j = r$ and the fact $\tau = \frac{\tau_0}{\lambda}$ that $\widehat{R}_r$ satisfies the required properties in Proposition 3.2.

We also have to find Q''_r satisfying (3.7) and (3.8) such that

$$Q''_r P_0 + P_0 Q''_r = -\frac{1}{2}[S_r P_0 + P_0 S_r].$$

Since by Lemma 3.1, S_r depends only on $Q'_1, \dots, Q'_r$, $Q''_1, \dots, Q''_{r-1}$ which have been already determined, we may define $Q''_r = -\frac{1}{2}S_r$. We see by Lemma 3.1 that Q''_r obeys (3.7) and (3.8) if K is chosen to be much larger than the square of the implicit constant. Therefore we obtain (3.24) at rank $r+1$ with terms satisfying the corresponding estimates. This concludes the proof. □

4 Proof of the main theorem

For any given $N \in \mathbb{N}^*$, once one has conjugated the operator $i\partial_t - \Delta + V$ into $i\partial_t - \Delta + V'_N + R'_N$ with V'_N exactly commuting with the modified Laplacian $\tilde{\Delta}$ and R'_N essentially being a bounded linear operator from $L^2(\mathbb{T}^d)$ to $H^N(\mathbb{T}^d)$, which has already been done in the previous section when m is taken to be so large that $m\tau \gg N$, we need to invert the transformation in order to get an estimate for the solution of the original Cauchy problem. Moreover, we have to compute the norms of the operators in order to obtain logarithmic growth of Sobolev norms from the energy inequality. To realize this, we begin with the following lemma.

Lemma 4.1. *Let $m \in \mathbb{N}^*$ and assume $\overline{Q}_j \in \mathcal{L}^{-j}_\tau(M,\rho,\lambda,0)$, $j = 1,2,\dots,m$. Then there are sequences $P_j \in \mathcal{L}^{-j}_\tau(M,\rho,\lambda,0)$, $1 \le j \le m$, $T_j \in \mathcal{L}^{-j}_\tau(M,\rho,\lambda,0)$, $m+1 \le j \le 2m$, $R'_j \in \mathcal{R}^{-\infty}_j(M, \frac{1}{2\lambda})$, $2 \le j \le 2m$ such that*

$$(I + \overline{Q}_1 + \cdots + \overline{Q}_m)(I + P_1 + \cdots + P_m) = I + \sum_{j=m+1}^{2m} T_j + \sum_{j=2}^{2m} R'_j \tag{4.1}$$

with

$$\begin{aligned} \|P_j\|_{j,0} &\le \sum_{\ell=1}^{j} \sum_{\substack{j_1+\cdots+j_\ell=j \\ 1\le j_1,\dots,j_\ell\le m}} C_2^{\ell-1}\|\overline{Q}_{j_1}\|_{j_1,0}\dots\|\overline{Q}_{j_\ell}\|_{j_\ell,0}, \quad 1 \le j \le m, \\ \|T_j\|_{j,0} &\le \sum_{\ell=2}^{j} \sum_{\substack{j_1+\cdots+j_\ell=j \\ 1\le j_1,\dots,j_\ell\le m}} C_2^{\ell-1}\|\overline{Q}_{j_1}\|_{j_1,0}\dots\|\overline{Q}_{j_\ell}\|_{j_\ell,0}, \quad m+1 \le j \le 2m, \\ |R'_j|_j &\le \sum_{\ell=2}^{j} \sum_{\substack{j_1+\cdots+j_\ell=j \\ 1\le j_1,\dots,j_\ell\le m}} C_2^{\ell-1}\|\overline{Q}_{j_1}\|_{j_1,0}\dots\|\overline{Q}_{j_\ell}\|_{j_\ell,0}, \quad 2 \le j \le 2m, \end{aligned} \tag{4.2}$$

where C_2 is an absolute constant.

Proof. Let $\overline{Q}_1, \dots, \overline{Q}_m$ be given. We set $P_1 = -\overline{Q}_1$ and by Proposition 2.7 we may recursively determine $P_j \in \mathcal{L}_\tau^{-j}(M, \rho, \lambda, 0)$ and $R'_j \in \mathcal{R}_j^{-\infty}(M, \frac{1}{2\lambda})$ for $j = 2, \dots, m$ such that

$$-\overline{Q}_j - \sum_{\substack{i+k=j \\ 1\le i,k\le m}} \overline{Q}_i P_k = P_j + R'_j \tag{4.3}$$

with

$$\begin{aligned} \|P_j\|_{j,0} &\lesssim \|\overline{Q}_j\|_{j,0} + \sum_{\substack{i+k=j \\ 1\le i,k\le m}} \|\overline{Q}_i\|_{i,0}\|P_k\|_{k,0}, \quad 2 \le j \le m, \\ |R'_j|_j &\lesssim \|\overline{Q}_j\|_{j,0} + \sum_{\substack{i+k=j \\ 1\le i,k\le m}} \|\overline{Q}_i\|_{i,0}\|P_k\|_{k,0}, \quad 2 \le j \le m. \end{aligned} \tag{4.4}$$

Consequently, we have

$$\begin{aligned} &(I+\overline{Q}_1 + \cdots + \overline{Q}_m)(I + P_1 + \cdots + P_m) \\ &= I + P_1 + \overline{Q}_1 + \sum_{j=2}^{m}\Big(P_j + \overline{Q}_j + \sum_{\substack{i+k=j \\ 1\le i,k\le m}} \overline{Q}_i P_k\Big) + \sum_{j=m+1}^{2m} \sum_{\substack{i+k=j \\ 1\le i,k\le m}} \overline{Q}_i P_k \\ &= I + \sum_{j=m+1}^{2m} \sum_{\substack{i+k=j \\ 1\le i,k\le m}} \overline{Q}_i P_k + \sum_{j=2}^{m} R'_j\,. \end{aligned} \tag{4.5}$$

Moreover by induction we obtain from (4.4) the required inequalities for P_j , $1 \le j \le m$ and the third inequality in (4.2) holds when $2 \le j \le m$, if C_2 is chosen to be larger than the implicit constant. Since $P_1, \dots, P_m$ have already been determined, by Proposition 2.7, we may also find $T_j \in \mathcal{L}_\tau^{-j}(M, \rho, \lambda, 0)$, $R'_j \in \mathcal{R}_j^{-\infty}(M, \frac{1}{2\lambda})$, $m + 1 \le j \le 2m$, such that

$$\sum_{\substack{i+k=j \\ 1\le i,k\le m}} \overline{Q}_i P_k = T_j + R'_j\,, \quad m + 1 \le j \le 2m, \tag{4.6}$$

with

$$\begin{aligned} \|T_j\|_{j,0} &\lesssim \sum_{\substack{i+k=j \\ 1\le i,k\le m}} \|\overline{Q}_i\|_{i,0}\|P_k\|_{k,0}, \quad m + 1 \le j \le 2m, \\ |R'_j|_j &\lesssim \sum_{\substack{i+k=j \\ 1\le i,k\le m}} \|\overline{Q}_i\|_{i,0}\|P_k\|_{k,0}, \quad m + 1 \le j \le 2m. \end{aligned} \tag{4.7}$$

Thus (4.1) follows by (4.5) and (4.6). The required estimates for T_j , R'_j , $m + 1 \le j \le 2m$, follow by (4.7) and the estimates of P_j , $1 \le j \le m$, which we have already obtained. This concludes the proof. □

Proof of the main theorem: Recall that $\tau = \frac{\tau_0}{\lambda} < \frac{1}{2\lambda}$, where τ_0 is given by Proposition 2.1 and λ given by (1.2). For any $N \in \mathbb{N}^*$, let m be an integer such that

$$N + 3 \le (m + 2)\tau < N + 4, \tag{4.8}$$

which implies

$$m > \frac{3}{\tau}\,, \quad m\tau > N. \tag{4.9}$$

Let the operators Q_j' , Q_j'', $1 \le j \le m$ be given by Proposition 3.2. Applying Lemma 4.1 to $\overline{Q}_1 = Q_1'$, $\overline{Q}_j = Q_j' + Q_{j-1}''$, $2 \le j \le m$, $\overline{Q}_{m+1} = Q_m''$, we may find

$$\begin{aligned}
&P_j \in \mathcal{L}_\tau^{-j}(M, \rho, \lambda, 0), \quad 1 \le j \le m+1,\\
&T_j \in \mathcal{L}_\tau^{-j}(M, \rho, \lambda, 0), \quad m+2 \le j \le 2m+2,\\
&R_j' \in \mathcal{R}_j^{-\infty}(M, \frac{1}{2\lambda}), \quad 2 \le j \le 2m+2
\end{aligned}$$

such that if we set $P^{m+1} = \sum_{j=1}^{m+1} P_j$, $Q^m = \sum_{j=1}^{m+1} \overline{Q}_j$

$$(I + Q^m)(I + P^{m+1}) = I + \sum_{j=m+2}^{2m+2} T_j + \sum_{j=2}^{2m+2} R_j'\,. \tag{4.10}$$

Moreover, (4.2) with m replaced by $m+1$ are satisfied by those operators. Since by (3.5), (3.7)

$$\|\overline{Q}_j\|_{j,0} \le \frac{2K^{j-\frac{1}{2}}}{j^2} h(\lambda, d)^j, \quad 1 \le j \le m+1, \tag{4.11}$$

we get by (4.2)

$$\begin{aligned}
\|P_j\|_{j,0} &\le \sum_{\ell=1}^{j} \sum_{j_1+\cdots+j_\ell=j} C_2^{\ell-1} \frac{2K^{j_1-\frac{1}{2}}}{j_1{}^2} \ldots \frac{2K^{j_\ell-\frac{1}{2}}}{j_\ell^2} h(\lambda, d)^j\\
&\le (2K)^j h(\lambda, d)^j, \quad 1 \le j \le m+1,\\
\|T_j\|_{j,0} &\le (2K)^j h(\lambda, d)^j, \quad m+2 \le j \le 2m+2,\\
|R_j'|_j &\le (2K)^j h(\lambda, d)^j, \quad 2 \le j \le 2m+2
\end{aligned} \tag{4.12}$$

if $K > (2C_2)^2$ and it is large enough so that Proposition 3.2 holds. For the solution u of (1.1), we set

$$v = (I + P^{m+1})u. \tag{4.13}$$

Then by Proposition 2.5, (4.12), for any $\sigma \in \mathbb{R}$,

$$\begin{aligned}
\|v(t)\|_{H^\sigma} &\lesssim \Big(1 + C_1^{|\sigma|} \sum_{j=1}^{m+1} \|P_j\|_{j,0} M^{j-1} [(j-1)!]^{\max(2,\mu)}\Big) \|u(t)\|_{H^\sigma}\\
&\lesssim C_1^{|\sigma|} (2KMh(\lambda, d))^{m+2} (m!)^{\max(2,\mu)} \|u(t)\|_{H^\sigma}.
\end{aligned} \tag{4.14}$$

Similarly, by Proposition 2.5, (4.12), for any $\sigma \in \mathbb{R}$,

$$\begin{aligned}
&\|\partial_t v(t)\|_{H^\sigma}\\
&\le \|\partial_t u(t)\|_{H^\sigma} + \sum_{j=1}^{m+1} \|[\partial_t, P_j]u(t)\|_{H^\sigma} + \sum_{j=1}^{m+1} \|P_j \partial_t u(t)\|_{H^\sigma}\\
&\lesssim C_1^{|\sigma|} (2KMh(\lambda, d))^{m+2} ((m+1)!)^{\max(2,\mu)} \|u(t)\|_{H^\sigma}\\
&\quad + C_1^{|\sigma|} (2KMh(\lambda, d))^{m+2} (m!)^{\max(2,\mu)} \|\partial_t u(t)\|_{H^\sigma},
\end{aligned} \tag{4.15}$$

and by (4.10), (4.13), (4.9), Proposition 2.5, Proposition 2.6, (4.11), (4.12)

$$\begin{aligned} &\|u(t)\|_{H^N} \\ &\leq \|(I+Q^m)v(t)\|_{H^N} + \sum_{j=m+2}^{2m+2} \|T_j\, u(t)\|_{H^{(m+2)\tau}} + \sum_{j=2}^{2m+2} \|R'_j\, u(t)\|_{H^{m\tau}} \\ &\lesssim (C_1KM)^{m+\frac{1}{2}} h(\lambda,d)^{m+1} (m!)^{\max(2,\mu)}\, \|v(t)\|_{H^N} \\ &\quad + (2KMh(\lambda,d))^{4m+3}[(2m+2)!]^{\max(2,\mu)+1}\, \|u(t)\|_{L^2}. \end{aligned} \tag{4.16}$$

By (3.9), (4.10) and (1.1)

$$(i\partial_t - \Delta + V^m)v = f + g, \tag{4.17}$$

where

$$\begin{aligned} f = &-\Big[\frac{1}{2}\sum_{j=m+1}^{2m+1}(S_jP_0+P_0S_j)v + \frac{1}{2}\sum_{j=1}^{2m+1}(R_jP_0+P_0R_j)v \\ &+ (\widetilde{S}_{m+1} + \sum_{j=m+1}^{2m+3}\overline{S}_j + \sum_{j=2}^{2m+3}\overline{R}_j + \sum_{j=1}^{m}\widehat{R}_j)\, v\Big], \end{aligned} \tag{4.18}$$

$$g = (I+Q^m)^*\Big[i\partial_t - \Delta + V, \sum_{j=m+2}^{2m+2} T_j + \sum_{j=2}^{2m+2} R'_j\Big]\, u. \tag{4.19}$$

Therefore by (2.5) and the property of V^m, we have

$$(i\partial_t - \Delta + V^m)(1-\tilde{\Delta})^{\frac{N}{2}}\, v = w,$$

where by Lemma 4.2 below

$$\|w\|_{L^2} \leq (4C_0C_1KMh(\lambda,d))^{5m+6}[(2m+3)!]^{2\max(2,\mu)}\|u_0\|_{L^2},$$

if K is in addition larger than the implicit constants of (4.24) and (4.25). Since V^m is self-adjoint, this implies the energy inequality

$$\begin{aligned} \|v(t)\|_{\widetilde{H}^N} &\leq \|v(0)\|_{\widetilde{H}^N} + \int_0^t (4C_0C_1KMh(\lambda,\, d))^{5m+6}[(2m+3)!]^{2\max(2,\mu)}\|u_0\|_{L^2}dt \\ &\leq \|v(0)\|_{\widetilde{H}^N} + |t|(4C_0C_1KMh(\lambda,\, d))^{5m+6}[(2m+3)!]^{2\max(2,\mu)}\|u_0\|_{L^2}. \end{aligned} \tag{4.20}$$

Now using (4.16), (2.7), (4.20), (4.14), the conservation law of the L^2-norm of (1.1) and (4.9), we deduce for some constant $C_{\lambda,d}$ independent of m and N

$$\|u(t)\|_{H^N} \leq C^N_{\lambda,d}[(2m+3)!]^{\frac{5}{2}\max(2,\mu)}(2+|t|)\|u_0\|_{H^N}, \tag{4.21}$$

if we use

$$(m!)^{\max(2,\mu)} \leq [(2m)!]^{\frac{1}{2}\max(2,\mu)}.$$

Since by (4.8), $2m+3 \leq ([\frac{10}{\tau}]+1)N$, we deduce from (4.21)

$$\|u(t)\|_{H^N} \leq C^N_{\lambda,d}[(([\frac{10}{\tau}]+1)N)!]^{\frac{5}{2}\max(2,\mu)}(2+|t|)\|u_0\|_{H^N}. \tag{4.22}$$

By Stirling's approximation $N! \sim \sqrt{2\pi N}(\frac{N}{e})^N$ for any $N \in \mathbb{N}$, there is a constant p_λ depend on τ and thus λ such that $(([\frac{10}{\tau}]+1)N)! \leq p^N_\lambda (N!)^{[\frac{10}{\tau}]+1}$, which, together with the fact that $\tau = \frac{\tau_0}{\lambda}$, allows us to rewrite (4.22) for some constant $\tilde{C}_{\lambda,d}$ independent of m, N, μ and for some constant ζ independent of m, N, μ and λ as

$$\|u(t)\|_{H^N} \leq \tilde{C}^N_{\lambda,d}(N!)^{\zeta\mu\lambda}(2+|t|)\|u_0\|_{H^N}. \tag{4.23}$$

Since (4.23) holds for any $N \in \mathbb{N}^*$, we deduce, for any $s > 0$, from the conservation law of the L^2-norm and interpolation

$$\|u(t)\|_{H^s} \leq \tilde{C}^{\theta N}_{\lambda,d}(N!)^{\zeta\mu\lambda\,\theta}(2+|t|)^{\theta}\|u_0\|_{H^s}$$

where θ satisfies $s = \theta N$, $\theta \in [0,1]$. Assuming $\|u_0\|_{H^s} \neq 0$, we obtain for any $N \in \mathbb{N}$ and for some other constant $C_{s,\lambda,d}$ independent of N

$$\Big(\frac{1}{C_{s,\lambda,d}}\Big(\frac{\|u(t)\|_{H^s}}{\|u_0\|_{H^s}}\Big)^{\frac{1}{\zeta\mu\lambda\,s}}\Big)^N \leq N!\,(2+|t|)^{\frac{1}{\zeta\mu\lambda}}.$$

This gives immediately for some other constant $C_{s,\lambda,d}$

$$\|u(t)\|_{H^s} \leq C_{s,\lambda,d}\big[\log(2+|t|)\big]^{\zeta\mu\lambda s}\|u_0\|_{H^s},$$

thus concludes the proof of the main theorem. □

Lemma 4.2. *Let f, g be the quantities defined respectively by (4.18) and (4.19). Then*

$$\|f\|_{\widetilde{H}^N} \lesssim (4C_0KMh(\lambda,\,d))^{5m+5}[(2m+3)!]^{2\max(2,\mu)}\|u_0\|_{L^2}, \tag{4.24}$$

$$\|g\|_{\widetilde{H}^N} \lesssim (2C_0C_1KMh(\lambda,\,d))^{5m+5}[(2m+3)!]^{2\max(2,\mu)}\|u_0\|_{L^2}, \tag{4.25}$$

where C_0 and C_1 are constants respectively defined by (2.7) *and* (2.14).

Proof. We have by (2.7), (4.8), the properties of S_j listed in Proposition 3.2, Proposition 2.4, Proposition 2.5, (4.15), (4.14), (1.1) and the conservation law of the L^2-norm of (1.1)

$$\begin{aligned}
&\|\sum_{j=m+1}^{2m+1}(S_j\,P_0 + P_0\,S_j)v(t)\|_{\widetilde{H}^N}\\
&\lesssim C_0^N\sum_{j=m+1}^{2m+1}\Big(\|S_j\,\partial_t v(t)\|_{H^{-2+(m+2)\tau}} + \|S_j\,\Delta v(t)\|_{H^{-2+(m+2)\tau}}\\
&\qquad\qquad + \|[i\partial_t, S_j]v(t)\|_{H^{(m+2)\tau}} + \|[\Delta, S_j]v(t)\|_{H^{(m+1)\tau}}\Big)\\
&\lesssim C_0^N\sum_{j=m+1}^{2m+1}\frac{K^j}{(j+1)^2}h(\lambda,\,d)^{j+1}M^j(j!)^{\max(2,\mu)}\Big(\|\partial_t v(t)\|_{H^{-2}} + \|\Delta v(t)\|_{H^{-2}}\Big)\\
&\qquad\qquad + C_0^N\sum_{j=m+1}^{2m+1}\frac{K^j}{(j+1)^2}h(\lambda,\,d)^{j+1}M^{j+1}[(j+1)!]^{\max(2,\mu)}\|v(t)\|_{L^2}\\
&\lesssim C_0^N(KM)^{2m+1}[(2m+1)!]^{\max(2,\mu)}h(\lambda,\,d)^{2m+2}\\
&\quad\times\Big((2KMh(\lambda,\,d))^{m+2}((m+1)!)^{\max(2,\mu)}\|u(t)\|_{L^2}\\
&\qquad\qquad + (2KMh(\lambda,\,d))^{m+2}(m!)^{\max(2,\mu)}\|\partial_t u(t)\|_{H^{-2}}\Big)\\
&\quad + C_0^N(KMh(\lambda,\,d))^{2m+2}[(2m+2)!]^{\max(2,\mu)}(2KMh(\lambda,\,d))^{m+2}(m!)^{\max(2,\mu)}\,\|u(t)\|_{L^2}\\
&\lesssim (2C_0KMh(\lambda,\,d))^{3m+4}[(2m+2)!]^{2\max(2,\mu)}\|u_0\|_{L^2}.
\end{aligned}$$

Using, in addition, (4.9) and Proposition 2.6, we similarly have

$$\|\sum_{j=1}^{2m+1}(R_jP_0+P_0R_j)v(t)\|_{\widetilde{H}^N}\lesssim(2C_0KMh(\lambda,\,d))^{5m+5}[(2m+3)!]^{2\max(2,\mu)}\|u_0\|_{L^2}.$$

By Proposition 2.5, Proposition 2.6 and Proposition 3.2, we easily deduce that the other terms in the the expression of f can be controlled by the right hand side of (4.24). We only need to point out that since $\overline{R}\in\mathcal{R}^{-\infty}_{j-1}(4M,\tau)$, when we use (2.15), M should be replaced by $4M$ and that is why in the right hand side of (4.24) the quantity '$4C_0KMh(\lambda,\,d)$' instead of '$2C_0KMh(\lambda,\,d)$' appears.

Next we want to show (4.25). First notice that by Proposition 2.5, (4.11), (4.9)

$$\begin{aligned}\|(I+Q^m)^*\|_{\mathcal{L}(H^N,H^N)}&\lesssim C_1^N\sum_{j=1}^{m+1}\frac{2K^{j-\frac{1}{2}}}{j^2}h(\lambda,\,d)^jM^{j-1}[(j-1)!]^{\max(2,\mu)}\\&\lesssim(C_1KM)^{m+\frac{1}{2}}h(\lambda,\,d)^{m+1}(m!)^{\max(2,\mu)}.\end{aligned}\tag{4.26}$$

On the other hand, by (2.7), (4.9), Proposition 2.5, Proposition 2.6, (4.12), the conservation law of the L^2-norm of (1.1),

$$\begin{aligned}&\|[i\partial_t,\sum_{j=m+2}^{2m+2}T_j+\sum_{j=2}^{2m+2}R'_j]u(t)\|_{\widetilde{H}^N}\\&\le C_0^N\sum_{j=m+2}^{2m+2}\|[i\partial_t,T_j]u(t)\|_{H^{m\tau}}+C_0^N\sum_{j=2}^{2m+2}\|[i\partial_t,R'_j]u(t)\|_{H^{m\tau}}\\&\lesssim C_0^N\sum_{j=m+2}^{2m+2}(2Kh(\lambda,\,d))^jM^j(j!)^{\max(2,\mu)}\|u(t)\|_{L^2}\\&\qquad+C_0^N\sum_{j=2}^{2m+2}(2Kh(\lambda,\,d))^jM^{2m+j+1}((j+1)!)^{\max(2,\mu)}(2m)!\|u(t)\|_{L^2}\\&\lesssim(2C_0KMh(\lambda,\,d))^{4m+4}[(2m+3)!]^{\max(2,\mu)+1}\|u_0\|_{L^2},\end{aligned}\tag{4.27}$$

and

$$\|[-\Delta,\sum_{j=m+2}^{2m+2}T_j+\sum_{j=2}^{2m+2}R'_j]u(t)\|_{\widetilde{H}^N}\lesssim(2C_0KMh(\lambda,\,d))^{4m+3}[(2m+2)!]^{\max(2,\mu)+1}\|u_0\|_{L^2}.\tag{4.28}$$

Since the quantity $\|[V,\sum_{j=m+2}^{2m+2}T_j+\sum_{j=2}^{2m+2}R'_j]u(t)\|_{\widetilde{H}^N}$ is also less than a constant times the last line of (4.27), by (4.26)-(4.28) we see that (4.25) holds true. □

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Résumé. Au cours des années récentes, plusieurs auteurs ont prouvé des résultats d'existence en temps grand pour des solutions d'équations de Klein-Gordon non-linéaires sur certaines variétés compactes, telles les sphères, lorsque les données initiales sont assez régulières et assez petites, et qu'un certain paramètre de masse évite un sous-ensemble de mesure nulle de la droite réelle. L'une des hypothèses fondamentales dans ces travaux est une propriété de séparation des valeurs propres du laplacien sur les variétés considérées. L'objet des deux premiers articles constituant cette thèse est d'examiner quels résultats peuvent être obtenus lorsqu'une telle hypothèse de séparation n'est plus vérifiée. Nous étudions le cas d'un opérateur de Klein-Gordon associé à l'oscillateur harmonique sur l'espace euclidien, et celui de l'opérateur de Klein-Gordon usuel sur le tore. Nous obtenons, par des méthodes de formes normales, des solutions existant sur des intervalles plus longs que ceux fournis par la théorie locale.
Le dernier article de cette thèse s'intéresse au problème de l'estimation en temps grand des normes Sobolev de solutions d'une équation de Schrödinger linéaire sur le tore, à potentiel dépendant du temps. Nous prouvons des bornes logarithmiques, lorsque le potentiel est Gevrey, généralisant des résultats antérieurs de Bourgain et Wang.

Mots clefs : Équation de Klein-Gordon non-linéaire ; Oscillateur harmonique ; Existence en temps grand ; Formes normales ; Équation de Schrödinger dépendant du temps ; Potentiel Gevrey ; Croissance des normes Sobolev.

Long-time existence and growth of Sobolev norms for solutions of semi-linear Klein-Gordon equations and linear Schrödinger equations on some manifolds

Abstract. In recent years, several authors proved long time existence results for solutions of non-linear Klein-Gordon equation on some compact manifolds, like spheres, when the initial data are smooth and small enough, and when some mass parameter avoids a subset of zero measure of the real line. One of the fundamental assumptions in these works is a separation property of the eigenvalues of the laplacian on the manifolds under consideration. The goal of the first two papers of this thesis is to examine which results may be obtained when such a separation assumption does not hold. We study two cases: a Klein-Gordon operator associated to the harmonic oscillator on the Euclidean space, and the usual Klein-Gordon operator on the torus. We get, using normal forms methods, solutions existing over longer time intervals than the ones given by the local theory.
The last paper of the thesis concerns long time estimates for Sobolev norms of solutions of a linear Schrödinger equation on the torus, with time dependent potential. We prove logarithmic bounds, when the potential is in the Gevrey class, extending results of Bourgain and Wang.

Keywords: Non-linear Klein-Gordon equation; Harmonic oscillator; Longtime existence; Normal form; Time dependent Schrödinger equation; Gevrey potential; Growth of Sobolev norms.

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